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# COMMON FIXED POINTS OF TWO ISOTONE MAPS ON A COMPLETE LATTICE

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## Introduction

Tarski's well-known result, that an isotone map on a complete lattice $L$ has a fixed point, was accompanied by the generalization that commuting isotone maps on a complete lattice have a common fixed point [4, pp. 288–289]. We give the proof here for a pair of isotone maps $S$ and $T$ (although the proof for a commuting family is essentially identical), as the basic definitions and the central argument will appear repeatedly throughout this paper.

Let $H = \{x \in L\colon Sx \leqslant x, Tx \leqslant x\}$. $H$ is non-empty since $1 \in H$, so let $h = \bigwedge\{x\colon x \in H\}$. If $x \in H$, $h \leqslant x$, and so $Sh \leqslant Sx$ by the isotony of $S$. Since $x \in H \Longrightarrow Sx \leqslant x$, we see that $Sh \leqslant x$. Taking the greatest lower bound of $H$ shows that $Sh \leqslant h$. We can similarly show that $Th \leqslant h$, and so $h \in H$. Since $S$ is isotone, $Sh \leqslant h \Longrightarrow S(Sh) \leqslant Sh$. Since $TS = ST$ and $Th \leqslant h$, $T(Sh) = S(Th) \leqslant Sh$. Therefore $Sh \in H$, and so $h \leqslant Sh$. Since both $Sh \leqslant h$ and $h \leqslant Sh$, we see that $Sh = h$. The identical argument also shows that $Th = h$.

Several well-known theorems on common fixed points, such as the Markov-Kakutani fixed point theorem ([2], p. 456), rely on commutativity. In the case of isotone maps on a complete lattice, however, it is possible to weaken substantially the hypothesis of commutativity, or to alter it, and still ensure the existence of a common fixed point.

A search of MathSciNet using the phrases

(1) complete lattices and simultaneous fixed points

(2) complete lattices and common fixed points

disclosed no papers covering the topics to be discussed here. As a result, it would not be unreasonable to presume that some of the problems encountered but not solved in this paper are open, and the last section will consist of a list of some of those questions.

Throughout this paper, $L$ will denote a complete lattice, and $S$ and $T$ isotone maps of $L$ into itself. Many of the results of this paper apply to families of isotone maps, rather than pairs, but a few of the results are most easily stated for pairs of maps, so for consistency of presentation we will remain within this framework.

An examination of the proof of Tarski's Theorem shows the reason that many results concerning pairs of isotone maps can be easily generalized to families. The proof essentially involves showing that, for each $Q$ in the family, both $Q(Qh) \leqslant Qh$ and $Q(Rh) \leqslant Rh$ for all $R$ different from $Q$ in the family. Proofs of simultaneous fixed points almost invariably proceed by showing that $Q(Rh) \leqslant Rh$ using only facts about $Q$ and $R$; even though all the other members of the family besides $R$ satisfy the hypotheses the proof looks only at a pair of maps rather that the entire family.

Broadly speaking, this paper concerns relations between $S$ and $T$ which will guarantee the existence of a common fixed point. These relations include algebraic equalities (of which commutativity is an example), lattice equalities, inequalities, and multiple equalities.

Most of the results presented require only algebraic or lattice hypotheses. However, it is known ([3]) that hypotheses related to continuity enable common fixed point theorems to be proved for commuting pairs of maps of the unit interval into itself; we shall later introduce similar hypotheses for isotone maps which will enable us to prove additional common fixed point theorems.

## I. Preliminaries

In the proof of Tarski's Theorem given previously, the definition of $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$ and $h = \bigwedge\{x\colon x \in H\}$ resulted in the conclusion that $Sh \leqslant h$ and $Th \leqslant h$, so $h \in H$. This idea, or its dual ($H = \{x\colon x \leqslant Sx,\ x \leqslant Tx\}$, $h = \bigvee\{x\colon x \in H\} \Rightarrow Sh \geqslant h$ and $Th \geqslant h$, so $h \in H$), will recur in every proof, so we shall merely define $H$, and use the conclusions just cited. We shall also use the results that $Sh \leqslant h \Longrightarrow S(Sh) \leqslant Sh$ and $Th \leqslant h \Longrightarrow T(Th) \leqslant Th$. Applying $S$ repeatedly to the inequality $Sh \leqslant h$ implies that $h \geqslant Sh \geqslant \ldots \geqslant S^n h \geqslant \ldots$, and similarly for $T$. Finally, once it has been established that $Sh \in H$ and $Th \in H$, the fact that $Sh = Th = h$ follows from the inequalities $Sh \leqslant h \leqslant Sh$ and $Th \leqslant h \leqslant Th$. Much of the effort in common fixed point proofs will be devoted to showing that $S(Th) \leqslant Th$ and/or $T(Sh) \leqslant Sh$.

We will also occasionally make use of the following facts about an isotone map $T$.

(1) $T\Big(\bigwedge\limits_{n=1}^{\infty} a_n\Big) \leqslant \bigwedge\limits_{n=1}^{\infty} Ta_n$.

(2) $T\Big(\bigvee_{n=1}^{\infty} a_n\Big) \geqslant \bigvee_{n=1}^{\infty} Ta_n$.

**Definition 1.** Let $I$ denote the identity map on $L$. Throughout this paper, $G$ will denote the semigroup under composition generated by $S, T$, and $I$.

**Lemma 1.** *Assume that $Q \in G$. If $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$, then $Qh \leqslant h$.*

Proof. We can assume that $Q$ is a word formed from the letters $S$, $T$, and $I$; the length of the word is the number of letters $S$ and $T$ in $Q$. Since the result is trivially true if $Q$ is either $S$, $T$, or $I$, we can assume the conclusion holds for words of length 0 ($I$ is the only such word) or 1 ($S$ and $T$ are the only such words). If the result is true for words of length $n$, then a word of length $n+1$ is either $SQ$ or $TQ$, where $Q$ is a word of length $n$. Since $S$ is isotone and $Qh \leqslant h$ by the inductive hypothesis, $SQh \leqslant Sh \leqslant h$; similarly $TQh \leqslant h$, establishing the desired result. $\square$

As the following example will show, the set of pairs $(S, T)$ such that $S$ and $T$ have a common fixed point possesses no easily-discernible algebraic or lattice structure. Let $L = \{0, a, 1\}$ with $0 < a < 1$ the only relations (of course, this is the lattice consisting of the integers 1, 2, and 3 with the usual order). Define operators $S$, $T$, $U$, $V$ as follows:

$$
\begin{aligned}
&S0 = Sa = S1 = 0,\\
&T0 = 0 \qquad Ta = T1 = 1,\\
&U0 = Ua = U1 = a,\\
&V0 = a, \qquad Va = V1 = 1.
\end{aligned}
$$

Note that 0 is a common fixed point for $S$ and $T$. Elementary computations show that $S \vee U = U$ and $T \vee U = V$, so $S \vee U$ and $T \vee U$ do not have a common fixed point. Since $SUx = 0$ and $TUx = 1$ for all $x \in L$, $SU$ and $TU$ do not have a common fixed point. Finally, $VS = U$ and $VT = V$, so $VS$ and $VT$ do not have a common fixed point.

However, it is possible to use existing common fixed point theorems to generate additional common fixed point theorems. We illustrate with a simple example, and then present a generalization which is probably well-known, but which the authors have been unable to locate in the literature.

Note that if $STx = Tx = x$, then $Sx = S(Tx) = x$, so $Sx = Tx = x$. This result is simply a matter of manipulating equalities. Now suppose that $S$ and $T$ are isotone maps of a complete lattice $L$ into itself such that $TST = ST^2$. The latter equality is simply a statement that the isotone maps $ST$ and $T$ commute, and therefore by Tarski's Theorem they have a common fixed point $x$. So $STx = Tx = x$, and by the result above, $Sx = Tx = x$.

This idea can be extended inductively. If $TSTST = ST^2ST$, then $ST$ and $TST$ commute, and so they have a common fixed point, which we denote by $x$. Therefore $TSTx = STx = x$. But then $Tx = T(STx) = x$, and as we have seen, $STx = Tx = x \Longrightarrow Sx = Tx = x$.

**Theorem 1.** *Let $X$ be a set, and let $F$, $G$, $U$, and $V$ be maps of $X$ into $X$. Let $\Phi(U,V)$ be an expression in $U$ and $V$ with the following property: if $\Phi(U,V)x = Vx = x$, then $Ux = x$. Let $\mathfrak{I}(U,V)$ be a set of conditions on $U$ and $V$ which ensure the existence of a common fixed point for $U$ and $V$.* (Note: In the example given just prior to this theorem, $\Phi(U,V)x = Vx = x$ is the equality $UVx = Vx = x$, and $\mathfrak{I}(U,V)$ is the condition that $U$ and $V$ commute.)

Let $Q_1 = F$, $Q_2 = G$, and define $Q_{n+1} = \Phi(Q_{n-1}, Q_n)$. If, for some integer $N$, the conditions $\mathfrak{I}(Q_{N+1}, Q_N)$ hold, then $F$ and $G$ have a common fixed point.

Proof. Since the conditions $\mathfrak{I}(Q_{N+1}, Q_N)$ hold, by assumption $\exists x \in X$ such that $Q_{N+1}x = Q_Nx = x$. Since $x = Q_Nx = Q_{N+1}x = \Phi(Q_{N-1}, Q_N)x$, we see that $Q_{N-1}x = x$ by the hypothesis on $\Phi$. So $Q_{N+1}x = Q_Nx = x \Longrightarrow Q_Nx = Q_{N-1} = x$. We can continue this procedure inductively down to $Q_3x = Q_2x = x$. But then $Q_3x = \Phi(Q_1, Q_2)x = Q_2x = x \Longrightarrow \Phi(F,G)x = Gx = x$. Therefore $Fx = x$. $\square$

Although Theorem 1 is formulated on an abstract set, the set may satisfy additional conditions which can be incorporated into the definition of both $\Phi$ and $\mathfrak{I}$.

As a simple example, if $L$ is a complete lattice, requiring that $I \wedge ST = T$ is sufficient to guarantee the existence of a common fixed point for $S$ and $T$. Let $H = \{x\colon Sx \leqslant x,\ Tx \leqslant x\}$. We see that, by Lemma 1, $STh = h \wedge STh = (I \wedge ST)h = Th$, so $S(Th) \leqslant Th$. Therefore (combining this with the always-known fact that $T(Th) \leqslant Th$), $Th \in H$, and so $Th = h$. So $STh = Th \Longrightarrow Sh = h$.

From Theorem 1, we therefore see that $I \wedge TST = ST$ is sufficient to guarantee the existence of a common fixed point for $S$ and $T$. The above result shows the existence of a common fixed point $x$ for $ST$ and $T$; then $STx = Tx = x$, which implies that $Sx = x$.

As we have stated, this paper will explore a wide range of conditions on $S$ and $T$ which are sufficient to guarantee the existence of a common fixed point. If there is a 'Holy Grail' in this area, it would be the ability to decide whether a specific equality relating $S$ and $T$ would guarantee the existence of a common fixed point without having to generate a proof dependent upon the specific equality. Although none of the proofs in this paper make use of deep theorems, some of the proofs require accurate definitions and precise computations. As a result, it would seem that a classification theorem of the type just described would be rather unlikely.

Algebraic conditions are relations between maps in $G$; commutativity ($ST = TS$) is a simple example of such a condition.

We start with a simple counterexample. Assume that $Q$ and $R$ are maps in $G$ such that at least one of the two, regarded as words in the letters $S$ and $T$, contains the letter $T$. Then no relation of the form $SQ = SR$ will guarantee the existence of a common fixed point. Let $L$ be the lattice $M_4$ (notation from [1], p. 4), which consists of 4 elements $\{0, 1, a, b\}$ with relations $0 < a < 1$ and $0 < b < 1$ (this is also the lattice of all subsets of $\{0, 1\}$). Define $Sx = a$ for all $x \in L$, and define $T0 = 0$, $T1 = 1$, $Ta = b$, and $Tb = a$. Then $SQ = SR = S$, but $S$ and $T$ have no common fixed points. We note in passing that $M_n$ can be used as a counterexample for many conjectures in this subject.

The simplest type of algebraic condition relating $S$ and $T$ is the equation $S^p = T^q$. It is possible to guarantee the existence of a common fixed point under weaker conditions.

**Theorem 2.** *Assume that for each $x \in L$ there exist integers $p = p(x)$, $q = q(x)$, $i = i(x)$, and $j = j(x)$ such that $S^p x \leqslant T^q x$ and $T^i x \leqslant S^j x$. Then $S$ and $T$ have a common fixed point.*

Proof. Let $H = \{x \in L\colon\ Sx \leqslant x, Tx \leqslant x\}$, and then let $u = \bigwedge\{Q^k h\colon\ Q = S, T; k = 0, 1, 2, \ldots\}$. We show that $Tu \leqslant u$.

Since $u \leqslant T^n h$ for any integer $n$, we see that $Tu \leqslant T^{n+1} h \leqslant T^n h$, so we must now show that $Tu \leqslant S^n h$ for any integer $n$.

By assumption, there exist integers $p_1$ and $q_1$ such that $T^{p_1} h \leqslant S^{q_1} h$. Having chosen $p_1 < \ldots < p_k$ and $q_1 < \ldots < q_k$ such that $T^{p_j} h \leqslant S^{q_j} h$ for $1 \leqslant j \leqslant k$, by assumption there exist integers $a$ and $b$ such that $T^a(T^{p_k} h) \leqslant S^b(T^{p_k} h)$. Let $p_{k+1} = p_k + a$ and $q_{k+1} = q_k + b$. Then $p_k < p_{k+1}$, $q_k < q_{k+1}$, and $T^{p_{k+1}} h = T^a(T^{p_k} h) \leqslant S^b(T^{p_k} h) \leqslant S^b(S^{q_k} h) = S^{q_{k+1}} h$.

Let $n \geqslant 0$. We show that $Tu \leqslant S^n h$. Choose $k$ such that $q_k \geqslant n$; then $u \leqslant T^{p_k - 1} h \implies Tu \leqslant T^{p_k} h \leqslant S^{q_k} h \leqslant S^n h$. Therefore $Tu \leqslant u$. A similar argument shows that $Su \leqslant u$, and so $u \in H$. Therefore $Th \leqslant h \leqslant u \leqslant Th$ and $Sh \leqslant h \leqslant u \leqslant Sh$, so $Th = Sh = h$. □

This type of argument will occur frequently throughout this paper. It is often easier to show that $Sh$ or $Th$ belongs to $H$ by defining an auxiliary element (such as $u$ in the above proof), and showing that $u$ belongs to $H$.

We can formalize the general idea. Suppose that $K$ is a subset of $L$ with the following properties:

(1) $Sh \in K$, $Th \in K$.

(2) For each $k \in K$, $\exists k_1 \in K$ such that $Sk_1 \leqslant k$.

(3) For each $k \in K$, $\exists k_2 \in K$ such that $Tk_2 \leqslant k$.

Then $Sh = Th = h$. If we let $u = \bigwedge\{k\colon\ k \in K\}$, then (2) $\Longrightarrow Su \leqslant u$ and (3) $\Longrightarrow Tu \leqslant u$, so $u \in H$. But then (1) $\Longrightarrow Sh \leqslant h \leqslant u \leqslant Sh$ and $Th \leqslant h \leqslant u \leqslant Th$, so $Sh = Th = h$.

The above argument demonstrates that the existence of $K$ is a sufficient condition for the existence of a common fixed point for $S$ and $T$, but it is not a necessary one. Let $L = \{0, 1, a, b, c\}$, where the relations are $0 < b < a < 1$ and $0 < c < a < 1$ (this lattice is dual to the one pictured on p. 131 of [1]). Define

$$S1 = 1, \qquad Sa = Sb = Sc = S0 = b,$$
$$T1 = 1, \qquad Ta = Tb = Tc = T0 = c.$$

$S$ and $T$ are isotone, $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\} = \{1, a\}$, and so $h = a$. 1 is a common fixed point for $S$ and $T$, but $h = a$ isn't, and all attempts to construct a set $K$ with the properties above are therefore doomed to fail.

Nonetheless, as will be demonstrated throughout this paper, the idea of constructing a set $K$ satisfying the three properties above is a fruitful one. One obvious candidate for $K$ is $\{Q^k h\colon\ Q = S, T;\ k = 0, 1, 2, \ldots\}$, which was utilized in Theorem 2. This set has the property that $S(S^n h) = S^{n+1}h \leqslant S^n h$ and $T(T^n h) = T^{n+1}h \leqslant T^n h$, so the easy part of the work is already done; we need only show that for each integer $n$ we can find elements $k = k(n)$ and $j = j(n)$ belonging to $k$ such that $Tk \leqslant S^n h$ and $Sj \leqslant T^n h$. A similar remark applies to another likely candidate, $\{Qh\colon\ Q \in G\}$.

Even an answer to a question such as, "Which equalities relating $S$ and $T$ guarantee that $\{Q^k h\colon\ Q = S, T; k = 0, 1, 2, \ldots\}$ satisfy properties (2) and (3) above?" would certainly be helpful in discovering when $S$ and $T$ have a common fixed point, and when they do not.

**Corollary 2.1.** *If either of the equalities $(ST)^k = T^n$ or $(ST)^k S = T^n$ holds, then $S$ and $T$ have a common fixed point.*

Proof. If $(ST)^k S = T^n$, then multiplying on the right by $T$ yields $(ST)^{k+1} = T^{n+1}$, so it suffices to show that the equality $(ST)^k = T^n$ ensures the existence of a common fixed point for $S$ and $T$. By Theorem 2, $ST$ and $T$ have a common fixed point $x$, and so $STx = Tx = x \Longrightarrow Sx = STx = x$. $\square$

Tarski's basic proof needs little modification to establish the following result.

**Theorem 3.** *Assume that for some integer $n$, $STS^n = T$. Then $S$ and $T$ have a common fixed point.*

P r o o f. Let $H = \{x \in L\colon Sx \leqslant x, Tx \leqslant x\}$. As usual, $S(Sh) \leqslant Sh$. Also $T(Sh) = STS^n(Sh) = STS^{n+1}h \leqslant Sh$ by Lemma 1, so $Sh \in H$, and therefore $Sh = h$. But then $Th = STS^n h = STh$, so $S(Th) \leqslant Th$. Since we always have $T(Th) \leqslant Th$, we see that $Th \in H$, and therefore $Th = h$. □

**Corollary 3.1.** *Assume that for some integer $n$, $STS^n = TS^p$, where $n > p$. Then $S$ and $T$ have a common fixed point.*

P r o o f. In Theorem 3, replace $T$ by $TS^p$. The hypothesis of the Corollary now becomes $S(TS^p)S^{n-p} = TS^p$, so $S$ and $TS^p$ have a common fixed point, which we denote by $x$. Using an argument familiar from Theorem 1, $x = TS^p x = Tx$. □

Tarski's basic proof can also be used to show that the equality $ST = TS^n$ guarantees the existence of a common fixed point. We shall later prove stronger results than this, and so will bypass this for the present.

Let $Q$, $R \in G$. Note that no equality of the form $S^n TQ = TR$ can guarantee the existence of common fixed points if $n > 1$. Let $M_{n+2}$ denote the lattice consisting of $\{0, 1, a_1, \ldots, a_n\}$, where the only relations are $0 < a_j < 1$ for $1 \leqslant j \leqslant n$. Let $\pi$ denote the cyclic permutation of the integers $\{1, \ldots, n\}$, and define $Tx = a_1$ for $x \in M_{n+2}$, $S0 = 0$, $S1 = 1$, $Sa_j = a_{\pi(j)}$ for $1 \leqslant j \leqslant n$. Then $S^n = I$, and since $TQ = TR = T$ we see that $T = IT = S^n TQ = TR$, but clearly $S$ and $T$ have no common fixed points.

However, under the right circumstances, two equalities of this type can ensure the existence of common fixed points.

**Theorem 4.** *Let $p$, $q$, and $r$ be positive integers, and assume that $GCD(p,q) = 1$. Suppose further that*

(1) $S^p T = TS$,

(2) $S^q T = TS^r$.

*Then $S$ and $T$ have a common fixed point.*

P r o o f. Let $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$. Note that $T(Sh) = S^p(Th) \leqslant S^p h \leqslant S^{p-1}h \leqslant \ldots \leqslant Sh$, and since $S(Sh) \leqslant Sh$ as usual, we see that $Sh \in H$. Therefore $Sh = h$.

We now prove that $S^{p^n} T^n h = T^n h$. For $n = 1$, $S^p Th = TSh = Th$. If it holds for $n$, then $S^{p^{n+1}} T^{n+1} h = (S^p)^{p^n} TT^n h = (S^p)^{p^n - 1} S^p TT^n h = (S^p)^{p^n - 1} TST^n h = (S^p)^{p^n - 2} S^p TST^n h = (S^p)^{p^n - 2} TS^2 T^n h = \ldots = TS^{p^n} T^n h = TT^n h = T^{n+1} h$.

Let $u = \bigwedge\limits_{n=0}^{\infty} \bigwedge\limits_{k=0}^{p^n - 1} S^k T^n h$.

Fix $n, k$ with $0 \leqslant k \leqslant p^n - 1$. If $1 \leqslant k \leqslant p^n - 1$, $u \leqslant S^{k-1} T^n h \Longrightarrow Su \leqslant S^k T^n h$. If $k = 0$, $u \leqslant S^{p^n - 1} T^n h \Longrightarrow Su \leqslant S^{p^n} T^n h = T^n h$. Since $Su \leqslant S^k T^n h$ for any $n, k$ with $0 \leqslant k \leqslant p^n - 1$, we see that $Su \leqslant u$.

We now show $Tu \leqslant u$. Since $GCD(p,q)=1$, $GCD(p^n,q)=1$ for any $n$. Fix $n,k$ with $0 \leqslant k \leqslant p^n - 1$. Since $GCD(p^n,q)=1$, choose $j$ such that $jq = k \pmod{p^n}$. Let $m$ be an integer such that $jq = k + mp^n$. Then $u \leqslant S^{jr}T^n h \Longrightarrow Tu \leqslant T(S^{jr}T^n h) = TS^r S^{(j-1)r}T^n h = S^q T S^{(j-1)r} T^n h = S^q T S^r S^{(j-2)r} T^n h = S^{2q} T S^{(j-2)r} T^n h = \ldots = S^{jq} T^{n+1} h \leqslant S^{jq} T^n h$, since $Th \leqslant h \Longrightarrow T^{n+1}h \leqslant T^n h$. So $Tu \leqslant S^{jq}T^n h = S^{k+mp^n}T^n h = S^{k+(m-1)p^n} S^{p^n} T^n h = S^{k+(m-1)p^n} T^n h = \ldots = S^k T^n h$. Since $Tu \leqslant S^k T^n h$ for any $n,k$ with $0 \leqslant k \leqslant p^n - 1$, we see that $Tu \leqslant u$.

So $u \in H$; therefore $h \leqslant u$. But $u \leqslant S^k T^n h$ for $0 \leqslant k \leqslant p^n - 1$. Letting $k = 0$ and $n = 1$, we see that $h \leqslant u \leqslant Th \leqslant h$, and so $Th = h$. □

The next two theorems involve symmetric conditions on $S$ and $T$.

**Theorem 5.** *Assume that for each $x \in L$, there exist integers $n = n(x)$ and $k = k(x)$ such that $ST^n x \leqslant T^n S x$ and $TS^k x \leqslant S^k T x$. Then $S$ and $T$ have a common fixed point.*

Proof. Let $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$. Choose $n$ such that $ST^n h \leqslant T^n S h$. Then $Th \leqslant h \Longrightarrow T^n(Th) \leqslant T^n h$, and so $T(T^n h) \leqslant T^n h$. Since $Sh \leqslant h$, $S(T^n h) \leqslant T^n(Sh) \leqslant T^n h$. Therefore $T^n h \in H$, and so $T^n h \leqslant \ldots \leqslant Th \leqslant h \leqslant T^n h \Longrightarrow Th = h$. This argument also shows that $Sh = h$. □

The following theorem also utilizes the idea of symmetric conditions.

**Theorem 6.** *Suppose that for each $x \in L$, there exist maps $Q = Q(X)$ and $R = R(x)$ belonging to $G$ such that $STx = TSQx$ and $TSx = STRx$. Then $S$ and $T$ have a common fixed point.*

Proof. This is just the Tarski proof combined with Lemma 1. Let $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$. Choose $Q = Q(h)$ such that $STh \leqslant TSQh$. Then $S(Th) = TSQh \leqslant Th$, since $SQ \in G$. Since $T(Th) \leqslant Th$ as usual, $Th \in H$, and so $Th = h$. Similarly, $Sh = h$. □

Theorems 2, 5, and 6 involve 'pointwise' hypotheses. Rather than require that $ST = TSQ$ and $TS = STR$ in Theorem 6, we allow the choice of $Q$ and $R$ to vary with $x$. Undoubtedly many of the theorems on common fixed points can be proved under 'pointwise' hypotheses. For instance, if one weakens the hypothesis of Theorem 3 to assume that for each $x \in L$ there is an integer $n = n(x)$ such that $STS^n x = Tx$ it is easy to see that the exact same proof can be used, because at the crucial moment one need only choose the integer $n = n(h)$ for the relevant equality.

In the previous section we investigated conditions in which the operators $S$ and $T$ were combined using composition of maps. The lattice structure of $L$ naturally imparts a lattice structure to the maps of $L$ into $L$, and in this section we investigate the effect of imposing lattice conditions on the maps $S$ and $T$.

In a complete lattice, it is possible to define inferior and superior limits.

$$\lim_{n\to\infty} \inf a_n = \bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} a_n,$$
$$\lim_{n\to\infty} \sup a_n = \bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} a_n.$$

We first investigate the existence of common fixed points when we have an equality of the form $\Phi(S) = \Psi(T)$, where $\Phi(S)$ is an expression from the complete lattice generated by the powers of $S$, and $\Psi(T)$ is an expression from the complete lattice generated by the powers of $T$.

**Theorem 7.** *Each of the equalities below ensures the existence of common fixed points for $S$ and $T$.*

(a) $\bigvee_{n=1}^{\infty} S^n = \bigvee_{n=1}^{\infty} T^n$,

(b) $\bigvee_{n=1}^{\infty} S^n = \bigwedge_{n=1}^{\infty} T^n$,

(c) $\bigvee_{n=1}^{\infty} S^n = \bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} T^n$,

(d) $\bigvee_{n=1}^{\infty} S^n = \bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} T^n$,

(e) $\bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} S^n = \bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} T^n$,

(f) $\bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} S^n = \bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} T^n$.

P r o o f. (a) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Then $Sh = \bigvee_{n=1}^{\infty} S^n h = \bigvee_{n=1}^{\infty} T^n h = Th$. Since $S(Sh) \leqslant Sh$ and $T(Sh) = T(Th) \leqslant Th = Sh$, we see that $Sh \in H$, and so $Sh = h$. Therefore $Th = Sh = h$ as well.

(b) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Then $Sh = \bigvee_{n=1}^{\infty} S^n h = \bigwedge_{n=1}^{\infty} T^n h$. Since $S(Sh) \leqslant Sh$ and $T(Sh) = T\Big(\bigwedge_{n=1}^{\infty} T^n h\Big) \leqslant \bigwedge_{n=1}^{\infty} T^{n+1} h = \bigwedge_{n=1}^{\infty} T^n h = \bigvee_{n=1}^{\infty} S^n h = Sh$. So $Sh \in H$, and therefore $Sh = h$.

Therefore $h = Sh = \bigwedge_{n=1}^{\infty} T^n h \leqslant Th \leqslant h$, and $Th = h$.

(c) and (d) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Then $\bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} T^n h = \bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} T^n h = \bigwedge_{n=1}^{\infty} T^n h$. In both (c) and (d), $Sh = \bigvee_{n=1}^{\infty} S^n h = \bigwedge_{n=1}^{\infty} T^n h$. So $S(Sh) \leqslant Sh$ and $T(Sh) = T\Bigl(\bigwedge_{n=1}^{\infty} T^n h\Bigr) \leqslant \bigwedge_{n=1}^{\infty} T^n h = Sh$. So $Sh \in H$, and therefore $Sh = h$. But then $h = Sh = \bigwedge_{n=1}^{\infty} T^n h \leqslant Th \leqslant h$, and so $Th = h$.

(e) and (f) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Then $\bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} T^n h = \bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} T^n h = \bigwedge_{n=1}^{\infty} T^n h$, and also $\bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} S^n h = \bigwedge_{n=1}^{\infty} S^n h$. Therefore, in both (e) and (f), $\bigwedge_{n=1}^{\infty} T^n h = \bigwedge_{n=1}^{\infty} S^n h$; denote this element by $u$. Then $u \leqslant S^n h \Longrightarrow Su \leqslant S^{n+1} h \leqslant S^n h \Longrightarrow Su \leqslant u$, and an analogous argument shows that $Tu \leqslant u$. So $u \in H$, and therefore both $Sh \leqslant h \leqslant u \leqslant Sh$ and $Th \leqslant h \leqslant u \leqslant Th$. So $Sh = Th = h$. □

The next few theorems give some indication of the extent to which the hypothesis of commutativity in Tarski's Theorem can be weakened.

**Theorem 8.** *Assume that*
(1) $ST \leqslant T\Bigl(\bigvee_{n=1}^{\infty} S^n\Bigr)$.
(2) *For each $x \in L$, there is an integer $n = n(x)$ such that $STx \geqslant TS^n x$.*
*Then $S$ and $T$ have a common fixed point.*

Proof. Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Since $S^n h \leqslant h$, $STh \leqslant T\Bigl(\bigvee_{n=1}^{\infty} S^n h\Bigr) \leqslant Th$. As usual, $T(Th) \leqslant Th$, so $Th \in H \Longrightarrow Th = h$.

Let $u = \bigwedge_{n=1}^{\infty} S^n h$. From assumption (2), $\exists n_1$ such that $TS^{n_1} h \leqslant STh = Sh$. Assume that $n_p > \ldots > n_1$ such that $TS^{n_p} h \leqslant S^p h$. Choose $k$ such that $TS^k(S^{n_p} h) \leqslant ST(S^{n_p} h)$, and let $n_{p+1} = n_p + k$. Then $TS^{n_{p+1}} h = TS^k(S^{n_p} h) \leqslant ST(S^{n_p} h) \leqslant S(S^p h) = S^{p+1} h$. Now $u \leqslant S^{n_p} h \Longrightarrow Tu \leqslant TS^{n_p} h \leqslant S^p h \Longrightarrow Tu \leqslant u$, and also $u \leqslant S^n h \Longrightarrow Su \leqslant S^{n+1} h \leqslant S^n h \Longrightarrow Su \leqslant u$. So $u \in H$, and therefore $Sh \leqslant h \leqslant u \leqslant Sh$. Therefore $Sh = h$, and so $h$ is a common fixed point for $S$ and $T$. □

From the standpoint of symmetry, as well as the improvement of Theorem 8, it would be nice to replace hypothesis (2) with the dual of hypothesis (1). In the next section we will introduce another condition which will enable us to achieve this result.

We are interested in discovering situations in which $F(S,T) = G(S,T)$ leads to common fixed points, where $F$ and $G$ are expressions (algebraic, lattice, or anything else) in $S$ and $T$.

**Theorem 9.** *Throughout this theorem, all meets and joins are taken over sets of integers, each of which contains* 1.

*Each of the following equalities ensures the existence of a common fixed point for* $S$ *and* $T$.

(a) $(\bigvee S^n)(\bigvee T^k) = (\bigvee T^k)(\bigvee S^n)$.

(b) $(\bigvee S^n)(\bigwedge T^k) = (\bigwedge T^k)(\bigvee S^n)$.

Proof. In both (a) and (b), let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Because of the restriction that 1 be a member of each index set, and since both $\{S^n h\}$ and $\{T^k h\}$ are descending chains, we see that $\bigvee S^n h = Sh$ and $\bigvee T^k h = Th$.

(a) $(\bigvee S^n)(\bigvee T^k)h = \bigvee S^n Th$; similarly $(\bigvee T^k)(\bigvee S^n)h = \bigvee T^k Sh$. So $\bigvee S^n Th = \bigvee T^k Sh$. Therefore $STh \leqslant \bigvee S^n Th = \bigvee T^k Sh \leqslant \bigvee T^k h = Th$. As usual, $T(Th) \leqslant Th$, so $Th \in H$, and therefore $Th = h$. The proof that $Sh = h$ just interchanges both the letters $S$ and $T$, and $n$ and $k$.

(b) Since $\bigvee S^n h = Sh$, we see that $(\bigvee S^n)(\bigwedge T^k h) = \bigwedge T^k Sh$. So $S(\bigwedge T^k h) \leqslant (\bigvee S^n)(\bigwedge T^k h) = \bigwedge T^k Sh \leqslant \bigwedge T^k h$. Since $T(\bigwedge T^k h) \leqslant \bigwedge T^{k+1} h = \bigwedge T^k h$, therefore $u = \bigwedge T^k h \in H$, and so $Th \leqslant h \leqslant u \leqslant Th$. So $Th = h$.

As a result, $Sh = (\bigvee S^n h) = (\bigvee S^n)(\bigwedge T^k h) = \bigwedge T^k Sh$. So $Sh \leqslant TSh$, and applying $T$ repeatedly to this inequality yields $Sh \leqslant TSh \leqslant T^2 Sh \leqslant \ldots$. Therefore $\bigwedge T^k Sh = TSh$, and so $Sh = TSh$. Since both $T(Sh) = Sh \leqslant Sh$ and $S(Sh) \leqslant Sh$ as usual, we see that $Sh \in H$. Therefore $Sh = h$. $\square$

**Theorem 10.** *Let* $Q, R$ *be isotone maps of* $L$ *into* $L$ *such that* $Q \geqslant \bigwedge_{n=1}^{\infty} S^n$ *and* $R \geqslant T(\bigwedge\{U\colon\ U \in G\})$. *Assume that* $N$ *is a positive integer such that* $ST \wedge Q = TS^N \wedge R$. *Then* $S$ *and* $T$ *have a common fixed point.*

Proof. Let $H = \{x\colon\ Sx \geqslant x, Tx \geqslant x\}$. Since $S^n h \geqslant Sh$ for all $n$, $Qh \geqslant Sh$. Also $Th \geqslant h \Longrightarrow STh \geqslant Sh$. Since $U \in G \Longrightarrow Uh \geqslant h$ by Lemma 1, we see that $Rh \geqslant Th$. So $TS^N h \wedge Rh = STh \wedge Qh \geqslant Sh \Longrightarrow TS^N h \geqslant Sh$.

Assume that $TS^{kN} h \geqslant S^k h$. Then $ST(S^{kN} h) \wedge QS^{kN} h = TS^N(S^{kN} h) \wedge RS^{kN} h$. But $ST(S^{kN} h) \geqslant S(S^k h) = S^{k+1} h$, and for each $n$, $S^n(S^{kN} h) \geqslant S^{k+1} h$, so $Q \geqslant \bigwedge_{n=1}^{\infty} S^n \Longrightarrow Q(S^{kN} h) \geqslant S^{k+1} h$. Therefore $S^{k+1} h \leqslant TS^N(S^{kN} h) \wedge RS^{kN} h \Longrightarrow TS^{(k+1)N} h \geqslant S^{k+1} h$.

Let $u = \bigvee_{n=1}^{\infty} S^n h$. Then $u \geqslant S^{kN}h \Longrightarrow Tu \geqslant TS^{kN}h \geqslant S^k h$ for all $k$, so $Tu \geqslant u$. Since $u \geqslant S^k h \Longrightarrow Su \geqslant S^{k+1}h \geqslant S^k h$, we see that $Su \geqslant u$. So $u \in H$. Therefore $Sh \geqslant h \geqslant u \geqslant Sh \Longrightarrow Sh = h$.

So $STh \wedge Qh = TS^N h \wedge Rh = Th \wedge Rh$. But $U \in G \Longrightarrow Uh \geqslant h$ by Lemma 1, and so $\bigwedge\{Uh\colon U \in G\} \geqslant h$. Therefore $Rh \geqslant Th$, and so $Th \wedge Rh = Th$. Consequently $STh \wedge Qh = Th$, which implies that $S(Th) \geqslant Th$. Since $T(Th) \geqslant Th$ as usual, we see that $Th \in H$, and so $Th = h$, completing the proof. $\square$

The following theorem investigates the result of changing the equality $ST \wedge Q = TS^N \wedge R$ to $ST \vee Q = TS^N \wedge R$.

**Theorem 11.** *Assume that there exist an integer $N$ and maps $Q$ and $R$ (not necessarily isotone) of $L$ into $L$ such that for each $x \in L$ there exist integers $j = j(x)$ and $k = k(x)$ with $0 \leqslant k \leqslant N$ and $j \geqslant 1$ such that $Qx \leqslant S^j x$, $Rx \geqslant TS^k x$. Assume also that $ST \vee Q = TS^N \wedge R$. Then $S$ and $T$ have a common fixed point.*

Proof. Let $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$. Choose $j$ and $k$ subject to the above restrictions such that $Qh \leqslant S^j h$, $Rh \geqslant TS^k h$. Then $(Q \vee ST)h = Qh \vee STh = (TS^N \wedge R)h = TS^N h \wedge Rh$. So $STh \leqslant TS^N h \leqslant Th$, since $S^N h \leqslant h$. As usual, $T(Th) \leqslant Th$, so $Th \in H$, and therefore $Th = h$.

So $(Q \vee ST)h = Qh \vee STh = Qh \vee Sh = Sh$, since $Qh \leqslant S^j h \leqslant Sh$. Also $(TS^N \wedge R)h = TS^N h \wedge Rh = TS^N h$, since $Rh \geqslant TS^k h$. Therefore $TS^N h = Sh$.

Assume inductively that $TS^{iN}h = S^i h$. Choose $j \geqslant 1$ such that $Q(S^{iN}h) \leqslant S^j(S^{iN}h) = S^{j+iN}h$ and $k$ with $0 \leqslant k \leqslant N$ such that $R(S^{iN}h) \geqslant TS^k(S^{iN}h) = TS^{k+iN}h$. Then $(Q \vee ST)(S^{iN}h) = Q(S^{iN}h) \vee STS^{iN}h = Q(S^{iN}h) \vee S^{i+1}h = S^{i+1}h$. Also $(TS^N \wedge R)(S^{iN}h) = TS^{(i+1)N}h \wedge RS^{iN}h = TS^{(i+1)N}h$, since $RS^{iN}h \geqslant TS^{k+iN}h \geqslant TS^{N+iN}h = TS^{(i+1)N}h$. So we can conclude by induction that $TS^{iN}h = S^i h$.

Let $u = \bigwedge_{i=1}^{\infty} S^i h$. Then $u \leqslant S^i h \Longrightarrow Su \leqslant S^{i+1}h \leqslant S^i h$, so $Su \leqslant u$. Also $u \leqslant S^{iN}h \Longrightarrow Tu \leqslant TS^{iN}h = S^i h$, so $Tu \leqslant u$. Therefore $u \in H$, and so $Sh \leqslant h \leqslant u \leqslant Sh$. So $Sh = h = Th$. $\square$

The following result is similar to Theorem 11, but does not appear to be a direct consequence of it.

**Theorem 12.** *Each of the following conditions the existence of a common fixed points for $S$ and $T$.*

(a) $(S \vee T)S = (S \wedge T)T$.

(b) $S(S \vee T) = T(S \wedge T)$.

P r o o f. (a) Let $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$. Since $(S \vee T)Sh = S^2h \vee TSh$ and $(S \wedge T)Th = STh \wedge T^2h$, we see that $S^2h \vee TSh = STh \wedge T^2h$. So $TSh \leqslant STh \leqslant Sh$, since $Th \leqslant h$. As usual, $S(Sh) \leqslant Sh$, and so $Sh \in H$. Therefore $Sh = h$.

So $S^2h \vee TSh = h \vee Th = h$, and therefore $h = S^2h \vee TSh = STh \wedge T^2h$. So $h \leqslant T^2h \leqslant Th \leqslant h$, and therefore $Th = h$.

(b) Let $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$. Note that $S(S \vee T)h = S(Sh \vee Th) \geqslant S^2h \vee STh$ and $T(S \wedge T)h = T(Sh \wedge Th) \leqslant TSh \wedge T^2h$. Since $Sh \leqslant h$, $STh \leqslant TSh \leqslant Th$. We also have $T(Th) \leqslant Th$, so $Th \in H$. Therefore $Th = h$.

Consequently, $S(S \vee T)h = S(Sh \vee Th) = S(Sh \vee h) = Sh$, and $T(S \wedge T)h = T(Sh \wedge Th) = T(Sh \wedge h) = TSh$. So $Sh = TSh$. Assume inductively that $TS^nh = S^nh$. Then $S(S \vee T)(S^nh) = S(S^{n+1}h \vee TS^nh) = S(S^{n+1}h \vee S^nh) = S^{n+1}h$, and $T(S \wedge T)(S^nh) = T(S^{n+1}h \wedge TS^nh) = T(S^{n+1}h \wedge S^nh) = TS^{n+1}h$. We conclude by induction that $TS^nh = S^nh$ for all $n$.

Let $u = \bigwedge\limits_{n=1}^{\infty} S^nh$. Since $u \leqslant S^nh$, $Su \leqslant S^{n+1}h \leqslant S^nh$ for any $n$, and so $Su \leqslant u$. Since $u \leqslant S^nh$, we also have $Tu \leqslant TS^nh = S^nh$ for any $n$, and so $Tu \leqslant u$. Therefore $u \in H$. So $Sh \leqslant h \leqslant u \leqslant Sh$, and so $Sh = h$, completing the proof. □

The identity map I does not enter the hypotheses in Theorems 8 through 12. Nonetheless, there are numerous common fixed point theorems which may be proved in which I appears. An example of this was given after Theorem 1. The following theorem is neither exhaustive nor best possible, but simply gives an idea of the types of equalities involving I that can guarantee the existence of common fixed points.

**Theorem 13.** *Each of the following equalities ensures the existence of common fixed points for $S$ and $T$.*

(a) $I \wedge ST = I \vee S$.
(b) $I \wedge ST = S \vee T$.
(c) $I \wedge ST = S \vee TS$.
(d) $I \wedge ST = T \vee TS$.
(e) $I \wedge ST = T \wedge TS$.
(f) $(I \vee S) \wedge T = (I \wedge S) \vee T$.
(g) $S(I \wedge T) = T(I \wedge S)$.
(h) $S(I \wedge T) = T(I \vee S)$.
(i) $(I \wedge T)S = (I \vee S)T$.

P r o o f. (a) let $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$. Note that $STh = h \wedge STh = (I \wedge ST)h = (I \vee S)h = h \vee Sh = h$. But $Th \leqslant h \Longrightarrow h = STh \leqslant Sh$, and combining this with the fact that $Sh \leqslant h$ shows that $Sh = h$. We also have $(I \wedge ST)(Th) = (I \vee S)(Th)$, so $Th \wedge ST^2h = Th \vee STh = Th \vee h = h$. This shows that $h \leqslant Th$, and since $Th \leqslant h$, we see that $Th = h$.

(b) Let $H = \{x\colon\ Sx \geqslant x, Tx \geqslant x\}$. Note that $Th \geqslant h \Longrightarrow STh \geqslant Sh \geqslant h$, so $h = (I \wedge ST)h = (S \vee T)h = Sh \vee Th$. Therefore $Sh \leqslant h \leqslant Sh$ and $Th \leqslant h \leqslant Th$, and so $Sh = Th = h$.

(c) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Note that $STh = h \wedge STh = (I \wedge ST)h = (S \vee TS)h = Sh \vee TSh$, so $TSh \leqslant STh$. But $Th \leqslant h \Longrightarrow STh \leqslant Sh$, and so $TSh \leqslant Sh$. As usual, $S(Sh) \leqslant Sh$, and so $Sh \in H$. Therefore $Sh = h$. So $STh = Sh \vee TSh = h \vee TSh = h$.

Now $Th \wedge ST^2h = (I \wedge ST)(Th) = (S \vee TS)(Th) = STh \vee T(STh) = STh \vee Th = h \vee Th = h$, so $h \leqslant Th \leqslant h \Longrightarrow Th = h$.

(d) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Note that $STh = h \wedge STh = (I \wedge ST)h = (T \vee TS)h = Th \vee TSh$. As in (c), $TSh \leqslant STh$, which leads to $Sh = h$. But then $Th = Th \vee Th = Th \vee TSh = STh$, and so $S(Th) \leqslant Th$. As usual, $T(Th) \leqslant Th$, and so $Th \in H$. Therefore $Th = h$.

(e) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Then $STh = h \wedge STh = (I \wedge ST)h = (T \wedge TS)h = Th \wedge TSh$. Therefore $STh \leqslant Th$. Combining this with $Th \leqslant h \Longrightarrow T(Th) \leqslant Th$, we see that $Th \in H$. Therefore $Th = h$.

Since $Sh = STh = Th \wedge TSh = h \wedge TSh = TSh$, we have $T(Sh) \leqslant Sh$. As usual, $S(Sh) \leqslant Sh$, and so $Sh \in H$. Therefore $Sh = h$.

(f) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Since $Sh \leqslant h$, $(I \vee S)h = h \vee Sh = h$, so $Th = h \wedge Th = [(I \vee S) \wedge T]h = [(I \wedge S) \vee T]h = (h \wedge Sh) \vee Th = Sh \vee Th$. So $Sh \leqslant Th$.

Note that $(I \vee S)(Sh) = Sh$, since $Sh \leqslant h \Longrightarrow S(Sh) \leqslant Sh$, and also $[(I \vee S) \wedge T](Sh) = [(I \wedge S) \vee T](Sh)$. Since $(I \wedge S)(Sh) = S^2h$, we see that $Sh \wedge T(Sh) = S^2h \vee T(Sh)$. Therefore $T(Sh) \leqslant Sh$. As usual, $S(Sh) \leqslant Sh$, and so $Sh \in H$. Therefore $Sh = h$. But then $h = Sh \leqslant Th \leqslant h \Longrightarrow Sh = Th = h$.

(g) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Since $Sh \leqslant h$, we see that both $(I \wedge S)h = Sh$ and $S(Sh) \leqslant Sh$. Similarly, $(I \wedge T)h = Th$ and $T(Th) \leqslant Th$. So $T(Sh) = T(I \wedge S)h = S(I \wedge T)h = S(Th) \leqslant Sh$, since $Th \leqslant h$, and so $Sh \in H$. Therefore $Sh = h$. Similarly, $S(Th) = S(I \wedge T)h = T(I \wedge S)h = T(Sh) \leqslant Th$, and so $Th \in H$. Therefore $Th = h$.

(h) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Note that $S(I \wedge T)h = S(h \wedge Th) = S(Th)$, since $Th \leqslant h$. Also, since $Sh \leqslant h$, we have $T(I \vee S)h = T(h \vee Sh) = Th$, so $STh = Th \leqslant Th$. As usual, $T(Th) \leqslant Th$, so $Th \in H$. Therefore $Th = h$. Substituting this into $STh = Th$, we see that $Sh = h$.

(i) Let $H = \{x\colon\ Sx \leqslant x, Tx \leqslant x\}$. Note that $(I \wedge T)(Sh) = Sh \wedge TSh$ and $(I \vee S)(Th) = Th \vee STh$, so $Th \vee STh = Sh \wedge TSh$. Therefore $STh \leqslant TSh \leqslant Th$, since $Sh \leqslant h$. Since $T(Th) \leqslant Th$, we see that $Th \in H$, and so $Th = h$.

Since $Th \vee STh = h \vee Sh = h$, we see that $h = Th \vee STh = Sh \wedge TSh$, so $h \leqslant Sh$. But $Sh \leqslant h$, and so $Sh = h$. $\square$

We now prove a theorem in which equalities involving commutation with inferior and superior limits ensures the existence of common fixed points.

**Theorem 14.** *Each of the following equalities ensures the existence of common fixed points for $S$ and $T$.*

(a) $S\Big(\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^n\Big)=\bigvee_{n=1}^{\infty}T^nS.$

(b) $S\Big(\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^n\Big)=\bigwedge_{n=1}^{\infty}T^nS.$

Proof. (a) Let $H=\{x\colon\ Sx\geqslant x, Tx\geqslant x\}$. Since $Th\leqslant\ldots\leqslant T^nh\leqslant\ldots$, we see that $\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nh=\bigvee_{n=1}^{\infty}T^nh$, and so $S(\bigvee_{n=1}^{\infty}T^nh)=\bigvee_{n=1}^{\infty}T^nSh\geqslant\bigvee_{n=1}^{\infty}T^nh$. Also, $T\Big(\bigvee_{n=1}^{\infty}T^nh\Big)\geqslant\bigvee_{n=1}^{\infty}T^{n+1}h=\bigvee_{n=1}^{\infty}T^nh$ since $Th\leqslant T^2h\leqslant\ldots\leqslant T^nh\leqslant\ldots$, so $\bigvee_{n=1}^{\infty}T^nh\in H$. Therefore $Th\leqslant\bigvee_{n=1}^{\infty}T^nh\leqslant h\leqslant Th$, and so $Th=h$.

Substituting $Th=h$ into the assumed equality, we obtain $Sh=\bigvee_{n=1}^{\infty}T^nSh$, and so $Sh\geqslant TSh$. Applying $T$ repeatedly to this inequality, we get $TSh\geqslant T^2Sh\geqslant\ldots\geqslant T^nSh\geqslant\ldots$, and so $TSh=\bigvee_{n=1}^{\infty}T^nSh$. Therefore $TSh=Sh\geqslant Sh$, and since $S(Sh)\geqslant Sh$ as usual, we see that $Sh\in H$. Therefore $Sh=h$.

(b) The proof here is essentially the same as that of (a). We define $H=\{x\colon\ Sx\leqslant x, Tx\leqslant x\}$, and use the identical arguments to show that $\bigwedge_{n=1}^{\infty}T^nh\in H$, which can be used to show that $Th=h$. This result is then substituted into the assumed equality to show that $Sh\in H$ in the same way as this was done in (a). $\square$

## IV. A continuity condition ensuring common fixed points

Assume that $a_1\geqslant a_2\geqslant\ldots\geqslant a_n\geqslant\ldots$. Although for any isotone map $T$ we have $T\Big(\bigwedge_{n=1}^{\infty}a_n\Big)\leqslant\bigwedge_{n=1}^{\infty}Ta_n$, in general equality need not hold. A simple example of this situation can be found on the unit interval $[0,1]$ by defining $T0=0$, $Tx=1$ if $x>0$. We say that $T$ is continuous from above when equality holds. An analogous definition can be made for increasing sequences; in which case we say $T$ is continuous from below.

This definition allows a significant strengthening of Theorem 8.

**Theorem 15.** *Let $S$ be continuous from above, and assume further that $T\Big(\bigwedge_{n=0}^{\infty}S^n\Big)\leqslant ST\leqslant T\Big(\bigvee_{n=0}^{\infty}S^n\Big)$. Then $S$ and $T$ have a common fixed point.*

P r o o f. Let $H=\{x\colon Sx\leqslant x, Tx\leqslant x\}$. Since $STh\leqslant T\Big(\bigvee_{n=0}^{\infty}S^nh\Big)\leqslant Th$, and $T(Th)\leqslant Th$ as usual, we see that $Th\in H$. So $Th=h$.

Therefore $Sh=STh\geqslant T\Big(\bigwedge_{n=0}^{\infty}S^nh\Big)$. Since $S$ is continuous from above, if we assume that $S^kh\geqslant T\Big(\bigwedge_{n=0}^{\infty}S^nh\Big)$, then $S^{k+1}h=S(S^kh)\geqslant ST\Big(\bigwedge_{n=0}^{\infty}S^nh\Big)\geqslant T\Big(\bigwedge_{j=0}^{\infty}S^j\Big(\bigwedge_{n=0}^{\infty}S^nh\Big)\Big)=T\Big(\bigwedge_{j=0}^{\infty}\bigwedge_{n=0}^{\infty}S^{j+n}h\Big)=T\Big(\bigwedge_{n=0}^{\infty}S^nh\Big)$, since $h\geqslant Sh\geqslant\ldots\geqslant S^nh\geqslant\ldots$. So $S^kh\geqslant T\Big(\bigwedge_{n=0}^{\infty}S^nh\Big)$ for all $k$, and therefore $\bigwedge_{n=0}^{\infty}S^nh\geqslant T\Big(\bigwedge_{n=0}^{\infty}S^nh\Big)$. Since $S$ is continuous from above, we also have $S\Big(\bigwedge_{n=0}^{\infty}S^nh\Big)=\bigwedge_{n=0}^{\infty}S^{n+1}h=\bigwedge_{n=0}^{\infty}S^nh$, and so $u=\bigwedge_{n=0}^{\infty}S^nh\in H$. Therefore $Sh\leqslant h\leqslant u\leqslant Sh$, and so $Sh=h$. $\square$

Assuming continuity from above also enables us to prove additional common fixed point theorems involving superior and inferior limits.

**Theorem 16.** *Assume that $T$ is continuous from above, and that*

$$S\Big(\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^n\Big)=\Big(\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^n\Big)S.$$

*Then $S$ and $T$ have a common fixed point.*

P r o o f. Let $H=\{x\colon Sx\leqslant x, Tx\leqslant x\}$. Since $Th\geqslant T^2h\geqslant\ldots\geqslant T^nh\geqslant\ldots$, we see that $\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nh=\bigvee_{N=1}^{\infty}\bigwedge_{n=1}^{\infty}T^nh=\bigwedge_{n=1}^{\infty}T^nh$. So $S\Big(\bigwedge_{n=1}^{\infty}T^nh\Big)=S\Big(\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nh\Big)=\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nSh\leqslant\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nh=\bigwedge_{n=1}^{\infty}T^nh$. Since we also have $T\Big(\bigwedge_{n=1}^{\infty}T^nh\Big)\leqslant\bigwedge_{n=1}^{\infty}T^{n+1}h=\bigwedge_{n=1}^{\infty}T^nh$, we see that $u=\bigwedge_{n=1}^{\infty}T^nh\in H$. Therefore $Th\leqslant h\leqslant u\leqslant Th$, so $Th=h$.

Therefore $Sh=S\Big(\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nh\Big)=\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nSh$. So, for any $N$, $Sh\geqslant\bigwedge_{n=N}^{\infty}T^nSh$. Since $T$ is continuous from above, we see that $TSh\geqslant T\Big(\bigwedge_{n=N}^{\infty}T^nSh\Big)=\bigwedge_{n=N}^{\infty}T^{n+1}Sh$, and so $TSh\geqslant\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^{n+1}Sh=\bigvee_{N=2}^{\infty}\bigwedge_{n=N}^{\infty}T^nSh$. Since $TSh=T^1Sh$, we have $TSh\geqslant\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nSh=Sh$. Applying $T$ repeatedly to $Sh\leqslant TSh$, we obtain $Sh\leqslant TSh\leqslant T^2Sh\leqslant\ldots\leqslant T^nSh\leqslant\ldots$. Therefore $Sh=\bigvee_{N=1}^{\infty}\bigwedge_{n=N}^{\infty}T^nSh=\bigvee_{N=1}^{\infty}T^NSh$. Since $Sh\leqslant TSh\leqslant T^2Sh\leqslant\ldots\leqslant T^NSh\leqslant\ldots$, we see that $Sh=$

$\bigvee_{N=1}^{\infty} T^N Sh \Longrightarrow Sh \geqslant TSh$. Since $S(Sh) \leqslant Sh$ as usual and $T(Sh) \leqslant Sh$, we see that $Sh \in H$. So $Sh = h$. □

The next result shows that Theorem 16 holds with the superior limit on the left side of the equality instead of the inferior limit.

**Theorem 17.** *Assume that $T$ is continuous from above, and that*

$$S\Big(\bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} T^n\Big) = \Big(\bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} T^n\Big) S.$$

*Then $S$ and $T$ have a common fixed point.*

Proof. Let $H = \{x\colon Sx \leqslant x, Tx \leqslant x\}$. Since $Th \geqslant T^2h \geqslant \ldots \geqslant T^nh \geqslant \ldots$, we see that $\bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} T^nh = \bigwedge_{N=1}^{\infty} T^Nh$. Consequently, $S\Big(\bigwedge_{n=1}^{\infty} T^nh\Big) = S\Big(\bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} T^nh\Big) = \bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} T^nSh \leqslant \bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} T^nh = \bigwedge_{n=1}^{\infty} T^nh$. Since we also have $T\Big(\bigwedge_{n=1}^{\infty} T^nh\Big) \leqslant \bigwedge_{n=1}^{\infty} T^{n+1}h = \bigwedge_{n=1}^{\infty} T^nh$, we see that $u = \bigwedge_{n=1}^{\infty} T^nh \in H$. Therefore $Th \leqslant h \leqslant u \leqslant Th$, so $Th = h$. The second paragraph of the proof of Theorem 16 now suffices to establish the theorem. □

## V. Open questions

Some of these may have easy answers that have eluded the authors; some, in particular the first two, may be fairly deep.

(1) Does there exist a rule which would enable one to decide when an algebraic equality relating $S$ and $T$ ensures the existence of a common fixed point, without the necessity of generating an equality-specific proof? Alternatively, can it be shown (possibly by metamathematical techniques) that no such rule exists?

(2) Let $L(T)$ be the complete lattice generated by the powers of $T$. Does there exist a rule which would enable one to decide when $Q \in L(T)$, $SQ = QS$ implies that $S$ and $T$ have a common point? Alternatively, can it be shown that no such rule exists?

(3) Do there exist other theorems in addition to Theorem 1 which provide ways to obtain additional common fixed point theorems from already-known results?

(4) Does the equality $ST^k = TS^n$ guarantee the existence of common fixed points, if $k > 1$ and $n > 1$?

(5) Does there exist a single equality relating three (or more) isotone maps which would guarantee a simultaneous fixed point for those maps?

(6) Can the hypothesis of continuity from above (or below) be eliminated from those theorems where it was used?

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