Czechoslovak Mathematical Journal, 49 (124) (1999), 843–847

COMMUTANTS AND DERIVATION RANGES

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(Received February 12, 1997)

Abstract. In this paper we obtain some results concerning the set $\mathcal{M} = \cup\{\overline{R(\delta_A)} \cap \{A\}'\colon A \in \mathcal{L}(\mathcal{H})\}$, where $\overline{R(\delta_A)}$ is the closure in the norm topology of the range of the inner derivation δ_A defined by $\delta_A(X) = AX - XA$. Here $\mathcal{H}$ stands for a Hilbert space and we prove that every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent if A is dominant, where $\overline{R(\delta_A)}^w$ is the closure of the range of δ_A in the weak topology.

Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space $\mathcal{H}$, the inner derivation induced by $A \in \mathcal{L}(\mathcal{H})$ being the map defined by

$$\delta_A\colon \mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{H}); \quad \delta_A(X) = AX - XA \quad (A \in \mathcal{L}(\mathcal{H})).$$

The identity is not a commutator, that is, $I \notin R(\delta_A)$ for any $A \in \mathcal{L}(\mathcal{H})$, where $R(\delta_A)$ denotes the range of δ_A. Nevertheless, J.H. Anderson in [2] proved the remarkable result that $I \in \overline{R(\delta_A)}$ for a large class of operators, where $\overline{R(\delta_A)}$ denotes the closure of the range of δ_A in the norm topology. This allowed him to define a new class of operators, called

$$J_A(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H})\colon I \in \overline{R(\delta_A)}\}.$$

Let $\mathcal{N} = \cup\Big\{R(\delta_A) \cap \{A\}'\colon A \in \mathcal{L}(\mathcal{H})\Big\}$, where $\{A\}'$ denotes the commutant of A. In finite dimension the set $\mathcal{N}$ is exactly the set of nilpotent operators, in infinite dimension the theorem of Kleïnecke-Shirokov [3] confirms that any operator in $\mathcal{N}$ is quasinilpotent. If we now consider instead of $\mathcal{N}$ the set

$$\mathcal{M} = \cup\Big\{\overline{R(\delta_A)} \cap \{A\}'\colon A \in \mathcal{L}(\mathcal{H})\Big\},$$

the theorem of Kleïneck-Shirokov can't be used. In other words an operator in $\mathcal{M}$ is not necessarily quasinilpotent; we can take as a counterexample the existence of an operator $A \in \mathcal{L}(\mathcal{H})$ such that $I \in \overline{R(\delta_A)}$.

J.H. Anderson [1, p. 135–136] proved that $\overline{R(\delta_A)} \cap \{A\}' = \{0\}$ if A is normal or isometric. Here we prove that any operator in $\mathcal{M}$ is nilpotent if $P(A)$ is normal, isometric or co-isometric for some polynomial P.

R.E. Weber [5] confirms that every compact operator in $\overline{R(\delta_A)}^w \cap \{A\}'$ is quasinilpotent, where $\overline{R(\delta_A)}^w$ is the weak closure of $R(\delta_A)$. If we now consider the set

$$\Big\{\overline{R(\delta_A)}^w \cap \{A^*\}' \colon\ A \in \mathcal{L}(\mathcal{H})\Big\},$$

we can ask: is every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ quasinilpotent? At this moment, we have not a global answer but we can partially answer this question with the assumption that A is dominant

Lemma 1. *Let A, $X \in \mathcal{L}(H)$, $T \in \{A\}'$ and $\varepsilon > 0$. If $\|A\| \leqslant 1$ and if $\|AX - XA - T\| < \varepsilon$, then for every $n \in \mathbb{N}$ we have*

$$\|(A^{n+1}X - XA^{n+1}) - (n+1)A^nT\| < (n+1)\varepsilon.$$

We recall that $\forall A \in \mathcal{L}(H)$, $\forall X \in \mathcal{L}(H)$ and $\forall T \in \{A\}'$ we have

$$A^nX - XA^n = nA^{n-1}T - \sum_{i=1}^{n} A^{n-i-1}(T - (AX - XA))A^i.$$

Proof. For $n = 0$ evident.

For $n = 1$ we have

$$A^2X - XA^2 = (A^2X - AXA) + (AXA - XA^2),$$

so,

$$\begin{aligned}\|(A^2X - XA^2) - 2AT\| &= \|(A^2X - AXA) - AT + (AXA - XA^2) - TA\| \\ &= \|A(AX - XA - T) + (AX - XA - T)A\| \\ &\leqslant 2\|A\|\|AX - XA - T\| < 2\varepsilon.\end{aligned}$$

Now suppose that for every $n \geqslant 2$ and for every $k \leqslant n$ we have

$$(*)\qquad \|(A^kX - XA^k) - kA^{k-1}T\| < k\varepsilon.$$

Since

$$(A^{n+1}X - X(A^{n+1}) - (n+1)A^nT) = A^n(AX - XA - T) + ((A^nX - XA^n) - nA^{n-1}T)A,$$

we have

$$\|(A^{n+1}X - X(A^{n+1}) - (n+1)A^nT\| < \varepsilon + n\varepsilon = (n+1)\varepsilon.$$

□

Theorem 2. *Let $A \in \mathcal{L}(\mathcal{H})$ and suppose that*

$$\overline{R(\delta_{P(A)})} \cap \{P(A)\}' = \{0\}$$

for some polynomial P, then every operator in $\overline{R(\delta_A)} \cap \{A\}'$ is nilpotent.

Proof. Let P be a polynomial of degree n and let $P^{(k)}$ be the k'th derivative of P. If

$$T \in \overline{R(\delta_A)} \cap \{A\}',$$

then there exists a sequence (X_n) in $\mathcal{L}(\mathcal{H})$ such that

$$AX_n - X_nA \to T;$$

since $T \in \{A\}'$ then

$$P^{(k)}(A)X_n - X_nP^{(k)}(A) \to P^{(k+1)}(A)T.$$

So

$$P(A)X_n - X_nP(A) \to P^{(1)}(A)T,$$

which shows that

$$P^{(1)}(A)T \in \overline{R(\delta_{P(A)})} \cap \{P(A)\}',$$

that is, $P^{(1)}(A)T = 0$. Also we have

$$P^{(1)}(A)X_n - X_nP^{(1)}(A) \to P^{(2)}(A)T,$$

which gives

$$0 = TP^{(1)}(A)X_nT - TX_nP^{(1)}(A)T \to P^{(2)}(A)T^3,$$

that is, $P^{(2)}(A)T^3 = 0$. By repeating the same argument it follows that $T^k = 0$ for a given integer number k, so T is nilpotent. In particular, every normal operator in $\overline{R(\delta_A)} \cap \{A\}'$ vanishes. □

Corollary 3. *Let $A \in \mathcal{L}(\mathcal{H})$. If $P(A)$ is normal, isometric or co-isometric ($AA^* = I$ or $A^*A = I$) for some polynomial P, then $\overline{R(\delta_A)} \cap \{A\}'$ is nilpotent.*

P r o o f. In [1, p. 136–137] Anderson showed that

$$\overline{R(\delta_{P(A)})} \cap \{P(A)\}' = \{0\}.$$

□

Definition 4. An operator $A \in \mathcal{L}(\mathcal{H})$ is called *dominant* if, for all complex λ, range$(A-\lambda) \subseteq$ range$(A-\lambda)^*$, or equivalently, if there is a real number $M_\lambda \geqslant 1$ such that

$$\|(A-\lambda)^* f\| \leqslant M_\lambda \|(A-\lambda) f\|$$

for all f in $\mathcal{H}$. If there is a constant M such that $M_\lambda \leqslant M$ for all λ, A is called *M-hyponormal*, and if $M = 1$, A is *hyponormal* (see [4]).

Theorem 5 [5]**.** *Let $A \in \mathcal{L}(\mathcal{H})$, then every compact operator in $\overline{R(\delta_A)}^w \cap \{A\}'$ is quasinilpotent.*

Theorem 6. *If $B \in \overline{R(\delta_A)}^w \cap \{A\}'$ and $f(B)$ is compact, where f is an analytic function on an open set containing $\sigma(A)$, then*

$$\sigma(B) \subset \{z\colon\ zf(z) = 0\}.$$

P r o o f. If $B \in \overline{R(\delta_A)}^w \cap \{A\}'$, then

$$AX_\alpha - X_\alpha A \xrightarrow{\text{w}} B;$$

since $f(B) \in \{A\}'$ we have

$$AX_\alpha f(B) - X_\alpha A f(B) \xrightarrow{\text{w}} Bf(B),$$

hence

$$AX_\alpha f(B) - X_\alpha f(B) A \xrightarrow{\text{w}} Bf(B),$$

that is,

$$Bf(B) \in \overline{R(\delta_A)}^w \cap \{A\}'.$$

Since $Bf(B)$ is compact, then $\sigma(Bf(B)) = g(\sigma(B)) = 0$ by Theorem 5, where $g(z) = zf(z)$. In particular, if $P(B)$ is compact for some polynomial P, then

$$\sigma(B) \subset \{z\colon\ zP(z) = 0\}.$$

□

Theorem 7. *Let A or A^* be a dominant operator.*
If $B \in \overline{R(\delta_A)}^w \cap \{A^*\}'$, then

$$\{\lambda \in \sigma_p(B^*) \colon \dim\ker(B^* - \overline{\lambda}) < \infty\} \subset \{0\}$$

or,

$$\{\lambda \in \sigma_p(B) \colon \dim\ker(B - \lambda) < \infty\} \subset \{0\},$$

where $\sigma_p(A)$ is the point spectrum of A.

P r o o f. Suppose that A is dominant and $B \in \overline{R(\delta_A)}^w \cap \{A^*\}'$, then

$$B^* \in \overline{R(\delta_{A^*})}^w \cap \{A\}'.$$

Let $\lambda \in \sigma_p(B^*)$ be such that $E = \ker(B^* - \lambda)$ is finite dimensional.

The subspace E is invariant under B^* and A. It is easy to verify that $A|_E$ is dominant, hence $A|_E$ is normal and so E reduces A (see [4]).

Let $H = E \oplus E^{\perp}$, then we can write

$$A = \begin{pmatrix} C & 0 \\ 0 & * \end{pmatrix}, \quad B^* = \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}.$$

Since $B^* \in \overline{R(\delta_{A^*})}^w$, then $\lambda I_E \in R(\delta_{C^*})$, and this necessarily implies $\lambda = 0$. □

By the same arguments as in the above proof we achieve the proof of the present theorem.

Corollary 8. *If A or A^* is a dominant operator, then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent.*

P r o o f. Suppose that $B \in \overline{R(\delta_A)}^w \cap \{A^*\}'$ with B compact and $\lambda \in \sigma(B) \setminus \{0\}$, then $\lambda \in \sigma_p(B)$ with $\dim\ker(B - \lambda) < \infty$ and $\overline{\lambda} \in \sigma_p(B^*)$ with $\dim\ker(B^* - \overline{\lambda}) < \infty$. It follows from Theorem 7 that B is quasinilpotent. □

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