ON THE KUROSH-ORE REPLACEMENT PROPERTY

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In this note we give a generalization of results of papers [2] and [3]. Moreover, it is shown here that the question (see [3], p. 183) has negative answer.

Next we recall some definitions and results. Let L be a lattice. Denote by $J(L)$ the set of all join-irreducible elements of L. For two elements $a, b \in L (a \leqslant b)$ we define $[a,b] = \{c \in L\colon\ a \leqslant c \leqslant b\}$. We say that a is a lower cover of b, if $a < b$ and $[a,b] = \{a,b\}$; in this case we write $a \prec b$. L is called strongly atomic (strongly dually atomic), if for any $a, b \in L$ with $a < b$ there is $p \in [a,b]$ such that $a \prec p (p \prec b)$.

Let L be a complete strongly dually atomic lattice. If $u \in J(L) - \{0\}$, then by u' we denote the uniquely determined lower cover of u. For $a \in L - \{1\}$, let $a^+ = \bigvee\{b \in L\colon\ a \prec b\}$. We say that L is locally modular, or locally distributive if for each $a \in L$, $a \neq 1$, the interval $[a, a^+]$ is a modular sublattice or a distributive sublattice, respectively.

A complete lattice L has replaceable irredundant $\vee$-decompositions ($\bigvee$-KORP, for short) if each element of L has at least one irredundant $\vee$-decomposition, and whenever $a = \bigvee T = \bigvee R$ are two irredundant $\vee$-decompositions of an element $a \in L$, for each $t \in T$ there exists $r \in R$ such that $a = \bigvee(T - \{t\}) \vee r$ is also an irredundant $\vee$-decomposition.

The $\bigwedge$-KORP is defined dually. P. Crawley and R. P. Dilworth investigated the $\bigwedge$-KORP in algebraic strongly atomic lattices. We recall the following result.

Theorem A (cf. [1], Theorems 7.5 and 7.6)**.** *Let L be an algebraic strongly atomic lattice.*

(i) *L has the $\bigwedge$-KORP if and only if for all $x, y \in L$, if the interval $[x, x \vee y]$ has exactly one atom, then the interval $[x \wedge y, y]$ has exactly one atom.*

(ii) *If L is semimodular, then L has the $\bigwedge$-KORP if and only if L is locally modular.*

Let us recall the following definition from [4], p. 570. A lattice L is strong if for $a, b \in L$, $u \in J(L)$, $b < u \leqslant a \vee b$ implies $u \leqslant a$. We note that for a class of strongly dually atomic lattices, the preceding notion of strongness is essentially the same as given in Stern [3]. The next result is a generalization of [3], Lemma 2.

Proposition 1. *A lattice L is strong if and only if L does not contain a pentagon isomorphic to the lattice in Figure 1 (where $u \in J(L)$).*

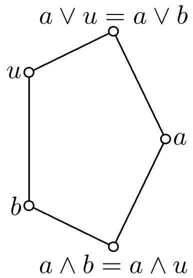

Fig. 1

Proof. Assume that L is not strong. Then there are $a, c \in L$, $u \in J(L)$ such that $c < u \leqslant a \vee c$ and $u \nleqslant a$. Let $b = c \vee (a \wedge u)$. Since u is join-irreducible, $b < u$. We have

$$a \wedge b \leqslant a \wedge u \leqslant a \wedge [c \vee (a \wedge u)] = a \wedge b,$$

and hence $a \wedge b = a \wedge u$. Now we observe that $a \wedge b < b$. Namely, $a \wedge b = b$ yields $b \leqslant a$ and thus $u \leqslant a \vee b = a$ contradicting our assumption $u \nleqslant a$. It is easy to see that $a \wedge b < a$ and $a < a \vee b \leqslant a \vee u$. On the other hand, $a \vee u \leqslant a \vee b$. Therefore, $a \vee b = a \vee u$, and thus L contains a pentagon isomorphic to the lattice in Figure 1.

The converse is trivial. $\square$

Remark 1. Proposition 1 implies that any modular is strong.

From Theorem 2 of [4] we obtain

Theorem B. *A semimodular, dually algebraic, strongly dually atomic lattice L has the $\bigvee$-KORP if and only if L is strong.*

We denote by $\mathcal{K}$ the class of all lattices L such that both L and its dual L^* are algebraic and strongly atomic.

The first major result is

Theorem 1. *Let $L \in \mathcal{K}$. If L is semimodular or lower semimodular, then L has both $\bigwedge$-KORP and the $\bigvee$-KORP if and only if L is modular.*

Proof. Without loss of generality we can assume that L is semimodular. Let L have both the $\bigwedge$-KORP and the $\bigvee$-KORP. We know that if an algebraic, strongly atomic lattice is both semimodular and lower semimodular, then it is modular (see [1], Theorem 3.6). Therefore, we only need to show that L is lower semimodular. Then we prove that L satisfies the following condition:

$$\text{(LS)} \qquad x \prec x \vee y \quad \text{implies} \quad x \wedge y \prec y.$$

Assume that $x \prec x \vee y$. By Theorem A(i), the interval $[x \wedge y, y]$ has exactly one atom, say p. We shall now prove that $p = y$. On the contrary, suppose that $p < y$. Since every element of L has at least one irredundant $\vee$-decomposition, we conclude that there is $u \in J(L)$ such that $u \leqslant y$ and $u \nleqslant p$. From Theorem B it follows that L is strong. We have

$$x \leqslant x \vee u' \leqslant x \vee y \quad \text{and} \quad x \prec x \vee y.$$

Observe that $x = x \vee u'$. Indeed, if $x \vee u' = x \vee y$, then $u \leqslant x \vee u'$ and strongness implies $u \leqslant x$, a contradiction. Therefore, $u' \leqslant x$. Hence, $u \wedge x \wedge y = u' \prec u$, and by semimodularity we deduce that $x \wedge y \prec u \vee (x \wedge y) \leqslant y$. Then $p = u \vee (x \wedge y)$, and this contradicts the fact that $u \nleqslant p$. Thus $x \wedge y \prec p = y$, that is, (LS) holds in L, and, in consequence, L is modular.

The converse follows from Theorems A and B. □

Remark 2. The preceding theorem generalizes Theorem 6 of [3], since any lattice satisfying the Descending Chain Condition is strongly atomic.

Theorem 2. *If $L \in \mathcal{K}$, then L has both the uniqueness property for irredundant $\wedge$-decompositions and the uniqueness property for irredundant $\vee$-decompositions if and only if L is distributive.*

The proof is the same as in [3], Theorem 7.

Theorem 3. *If $L \in \mathcal{K}$, then*

(i) *L is strong and locally modular if and only if L is modular.*

(ii) *L is strong and locally distributive if and only if L is distributive.*

P r o o f. (i). If L is locally modular, then L is also semimodular (see [1], p. 25). From Theorems A(ii) and B we conclude that L has both the $\bigwedge$-KORP and the $\bigvee$-KORP. Therefore, by Theorem 1, L is modular. The converse is obvious.

(ii). Let L be strong and locally distributive. By local distributivity, every modular sublattice of L is distributive (cf. [1], first paragraph on page 53). But L is modular, which follows from (i). Consequently, L is distributive.

The converse is clear. □

Remark 3. Theorem 3 is a generalization of Theorems 1 and 2 of [2].

Finally we recall that a complete lattice L has the Kurosh-Ore property for $\vee$-decompositions ($\bigvee$-KOP, for short), if every element of L has an irredundant finite $\vee$-decomposition and for each $a \in L$, the number of join-irreducible elements in any irredundant finite $\vee$-decomposition of a is unique. In a dual way one defines the $\bigwedge$-KOP. It is obvious that the KORP implies the corresponding KOP, whereas the converse does not hold in general. In semimodular algebraic lattices satisfying the Descending Chain Condition, the $\bigwedge$-KORP is equivalent to the $\bigwedge$-KOP (see [1], Theorems 7.6 and 7.7). Hence in Theorem 6 of [3] we may replace the $\bigwedge$-KORP by the $\bigwedge$-KOP, but here it is not possible to replace the $\bigvee$-KORP by the $\bigvee$-KOP, that is, the question of [3] has a negative answer. Indeed, let L be a lattice diagrammed in Figure 2. L is locally modular, and therefore it has the $\bigwedge$-KORP (and, evidently, the $\bigwedge$-KOP). This lattice also has the $\bigvee$-KOP, whereas the $\bigvee$-KORP does not hold.

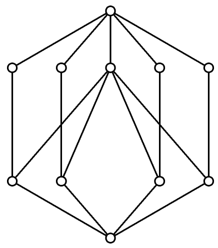

Fig. 2

References

[1] *P. Crawley, R. P. Dilworth*: Algebraic Theory of Lattices. Prentice Hall (Englewood Cliffs, New Jersey, 1973).

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