Czechoslovak Mathematical Journal, 49 (124) (1999), 825–830

EXTENDING n TIMES DIFFERENTIABLE FUNCTIONS OF SEVERAL VARIABLES

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(Received December 20, 1996)

Abstract. It is shown that n times Peano differentiable functions defined on a closed subset of $\mathbb{R}^m$ and satisfying a certain condition on that set can be extended to n times Peano differentiable functions defined on $\mathbb{R}^m$ if and only if the nth order Peano derivatives are Baire class one functions.

1. Introduction

The problem of extending differentiable functions of one variable was first attacked by V. Jarník in his 1923 paper [6]. There he showed that if a function, f, defined on a perfect subset, H, of $\mathbb{R}$ is differentiable everywhere on H relative to H, then it can be extended to a function differentiable on all of $\mathbb{R}$. Since H is perfect, it follows that the derivative of the extended function agrees with f' on H. His result was reestablished in 1974 by Petruska and Laczkovich in [8]. The result was improved in 1984 by Mařík in [7] where he succeeded in replacing "perfect" with "closed". In that paper Mařík showed by example that it wasn't possible to extend a function that is twice differentiable in the usual sense to a twice differentiable function on all of $\mathbb{R}$. However, in 1994 and 1997 it was shown that under some conditions on the function or on the set it is possible to extend a function n times Peano differentiable on a closed set to a function n times Peano differentiable on all of $\mathbb{R}$ whose Peano derivatives agree with those of the original function on H. (See [3] and [2].)

The study of the extension problem for functions of several variables was begun by Aversa, Laczkovich and Preiss in their 1985 paper [1]. There they showed that a function, f, of m variables which is differentiable on a closed subset of $\mathbb{R}^m$ can be extended to a function differentiable on all of $\mathbb{R}^m$ if and only if the derivative of f; that is, the gradient of f, is a function of Baire class one. The study is continued in this work where the case of a function which is n times Peano differentiable on a closed subset of $\mathbb{R}^m$ is investigated. It is shown that if the Peano derivatives satisfy a condition analogous to the one variable condition, then the extension problem has a positive solution if and only if the nth order Peano derivatives are of Baire class one. This result can be interpreted as the limiting case of the well-known Whitney Extension Theorem. (See [9].)

2. Dedication

The authors wish to dedicate this paper to the memory of Jan Mařík. Professor Mařík made significant contributions to several branches of analysis including some in the area of this paper. He is remembered with great fondness and is deeply missed by all of us who knew him.

3. Definitions and Notation

We adopt the usual conventions for working with differentiation in $\mathbb{R}^m$. By a multi-index, α, we mean an m-tuple, $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $\alpha_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \ldots, m$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, then $|\alpha| = \sum_{i=1}^{m} \alpha_i$ and $\alpha! = \prod_{i=1}^{m} \alpha_i!$. In addition for $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ by x^α we mean $\prod_{i=1}^{m} x_i^{\alpha_i}$. With this notation we are now in a position to define Peano differentiation.

Definition 3.1. Let $n \in \mathbb{N}$, let $H \subset \mathbb{R}^m$ be closed, let $f\colon H \to \mathbb{R}$ and let $x \in H$. Then f is n times Peano differentiable at x means for each multi-index, α, with $|\alpha| \leqslant n$ there is a number, $f_\alpha(x)$, such that for each $u \in H$,

$$f(u) = \sum_{0 \leqslant |\alpha| \leqslant n} \frac{f_\alpha(x)}{\alpha!}(u-x)^\alpha + o(\|u-x\|^n).$$

It should be noted that $f_{(0,0,\ldots,0)}(x) = f(x)$.

Some additional notation and a simple fact are also needed in the proof of the theorem to follow. If α and β are two multi-indices, then $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_m + \beta_m)$. Moreover $\beta \leqslant \alpha$ means $\alpha_i \leqslant \beta_i$ for each $i = 1, 2, \ldots, m$. For $\beta \leqslant \alpha$ we set $\alpha - \beta = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_m - \beta_m)$ and $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$. Using this notation the following version of the Binomial Theorem holds. If x and $y \in \mathbb{R}^m$, then $(x+y)^\alpha = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} x^\beta y^{\alpha-\beta}$.

Theorem 4.1. *Let H be a closed subset of $\mathbb{R}^m$ and let $f\colon H \to \mathbb{R}$ be n times Peano differentiable on H. Assume that for all $0 \leqslant |\alpha| \leqslant n-1$ and $0 \leqslant |\alpha+\gamma| \leqslant n$ we have that f_α is $|\gamma|$ times Peano differentiable on H with $(f_\alpha)_\gamma = f_{\alpha+\gamma}$. Then there is a function $F\colon \mathbb{R}^m \to \mathbb{R}$ that is n times Peano differentiable on $\mathbb{R}^m$ such that $F_\alpha = f_\alpha$ on H for all $|\alpha| \leqslant n$ if and only if for each α with $|\alpha| = n$ the function f_α is of Baire class one.*

Proof. If such a function, F, exists, then for each $0 \leqslant |\alpha| \leqslant n$ we have f_α is of Baire class one. (See [4].)

Assume that f_α is of Baire class one for all α with $|\alpha| = n$. For $u \in \mathbb{R}^m \setminus H$, let $t = t(u)$ denote a point in H such that $\operatorname{dist}(u, H) = \|t-u\|$. For $|\alpha| = n$ let g_α be defined on $\mathbb{R}^m \setminus H$ as follows. Since f_α is Baire 1, there is a sequence of Lipschitz functions $\{f_{\alpha,k}\}$ on H converging pointwise to f_α. (See [5].) Let $B_k \to \infty$ be an increasing sequence of positive real numbers with $\dfrac{\|f_{\alpha,k}(x) - f_{\alpha,k}(y)\|}{\|x-y\|} \leqslant B_k$ for all x, $y \in H$ and let

$$g_\alpha(u) = \begin{cases} f_{\alpha,k}(t(u)) & \text{if } \dfrac{1}{B_{k+1}^2} < \|t(u)-u\| \leqslant \dfrac{1}{B_k^2}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{\Phi_j\}$ be a locally finite, C^n partition of unity on $\mathbb{R}^m \setminus H$ subordinated to a collection of open balls with center $u_j \in \mathbb{R}^m \setminus H$ and radius $\frac{1}{2}\operatorname{dist}(u_j, H)$. For each j let $t_j = t(u_j)$. Let

$$F(u) = \begin{cases} \sum\limits_j \Phi_j(u)\Bigl(\sum\limits_{0\leqslant|\alpha|\leqslant n-1} (u-t_j)^\alpha \dfrac{f_\alpha(t_j)}{\alpha!} + \sum\limits_{|\alpha|=n} (u-t_j)^\alpha \dfrac{g_\alpha(u_j)}{\alpha!}\Bigr) & \text{for } u \notin H, \\ f(u) & \text{for } u \in H. \end{cases}$$

Since $\{\Phi_j\}$ is a C^n partition of unity, F is C^n on $\mathbb{R}^m \setminus H$. Consequently F is certainly n times Peano differentiable on $\mathbb{R}^m \setminus H$. To show that F is n times Peano differentiable on H, let $x \in H$. It suffices to consider only $u \notin H$. Set

$$D = F(u) - \sum_{0\leqslant|\alpha|\leqslant n} (u-x)^\alpha \frac{f_\alpha(x)}{\alpha!}.$$

Since $\sum_j \Phi_j(u) \equiv 1$,

$$
\begin{aligned}
D &= F(u) - \sum_j \Phi_j(u)\Bigg(\sum_{0\leqslant|\alpha|\leqslant n} (u-x)^\alpha \frac{f_\alpha x)}{\alpha!}\Bigg)\\
&= \sum_j \Phi_j(u)\Bigg(\sum_{0\leqslant|\alpha|\leqslant n-1} (u-t_j)^\alpha \frac{f_\alpha(t_j)}{\alpha!} + \sum_{|\alpha|=n} (u-t_j)^\alpha \frac{g_\alpha(u_j)}{\alpha!}\Bigg)\\
&\quad - \sum_j \Phi_j(u) \sum_{0\leqslant|\alpha|\leqslant n} (u-x)^\alpha \frac{f_\alpha(x)}{\alpha!}\\
&= \sum_j \Phi_j(u)\Bigg(\sum_{0\leqslant|\alpha|\leqslant n-1} (u-t_j)^\alpha \frac{f_\alpha(t_j)}{\alpha!} + \sum_{|\alpha|=n} (u-t_j)^\alpha \frac{g_\alpha(u_j)}{\alpha!}\Bigg)\\
&\quad - \sum_j \Phi_j(u) \sum_{0\leqslant|\alpha|\leqslant n} (u-t_j+t_j-x)^\alpha \frac{f_\alpha(x)}{\alpha!}\\
&= \sum_j \Phi_j(u)\Bigg(\sum_{0\leqslant|\alpha|\leqslant n-1} (u-t_j)^\alpha \frac{f_\alpha(t_j)}{\alpha!} + \sum_{|\alpha|=n} (u-t_j)^\alpha \frac{g_\alpha(u_j)}{\alpha!}\Bigg)\\
&\quad - \sum_j \Phi_j(u) \sum_{0\leqslant|\alpha|\leqslant n} \Bigg(\sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (u-t_j)^\beta (t_j-x)^{\alpha-\beta}\Bigg) \frac{f_\alpha(x)}{\alpha!}\\
&= \sum_j \Phi_j(u)\Bigg(\sum_{0\leqslant|\alpha|\leqslant n-1} (u-t_j)^\alpha \frac{f_\alpha(t_j)}{\alpha!} + \sum_{|\alpha|=n} (u-t_j)^\alpha \frac{g_\alpha(u_j)}{\alpha!}\Bigg)\\
&\quad - \sum_j \Phi_j(u)\Bigg(\sum_{0\leqslant|\alpha+\gamma|\leqslant n} \binom{\alpha+\gamma}{\alpha} (u-t_j)^\alpha (t_j-x)^\gamma \frac{f_{\alpha+\gamma}(x)}{(\alpha+\gamma)!}\Bigg)\\
&= \sum_j \Phi_j(u)\Bigg(\sum_{0\leqslant|\alpha|\leqslant n-1} (u-t_j)^\alpha \frac{f_\alpha(t_j)}{\alpha!} + \sum_{|\alpha|=n} (u-t_j)^\alpha \frac{g_\alpha(u_j)}{\alpha!}\Bigg)\\
&\quad - \sum_j \Phi_j(u)\Bigg(\sum_{0\leqslant|\alpha|\leqslant n} \sum_{0\leqslant|\gamma|\leqslant n-|\alpha|} (u-t_j)^\alpha (t_j-x)^\gamma \frac{f_{\alpha+\gamma}(x)}{\alpha!\gamma!}\Bigg)\\
&= \sum_j \Phi_j(u) \sum_{0\leqslant|\alpha|\leqslant n-1} \frac{(u-t_j)^\alpha}{\alpha!}\Bigg(f_\alpha(t_j) - \sum_{0\leqslant|\gamma|\leqslant n-|\alpha|} (t_j-x)^\gamma \frac{f_{\alpha+\gamma}(x)}{\gamma!}\Bigg)\\
&\quad + \sum_j \Phi_j(u)\Bigg(\sum_{|\alpha|=n} \frac{(u-t_j)^\alpha}{\alpha!}(g_\alpha(u_j) - f_\alpha(x))\Bigg).
\end{aligned}
$$

We show that each of these two sums is $o(\|u-x\|^n)$ as $u \to x$. By assumption for $0 \leqslant |\alpha| \leqslant n-1$ we have $f_\alpha(t) - \sum_{0\leqslant|\gamma|\leqslant n-|\alpha|} (t-x)^\gamma \frac{f_{\alpha+\gamma}(x)}{\gamma!} = o(\|t-x\|^{n-|\alpha|})$ for any

$t \in H$. Hence

$$\sum_{0 \leqslant |\alpha| \leqslant n-1} \frac{(u-t_j)^\alpha}{\alpha!}\left(f_\alpha(t_j) - \sum_{0 \leqslant |\gamma| \leqslant n-|\alpha|} (t_j - x)^\gamma \frac{f_{\alpha+\gamma}(x)}{\gamma!}\right) = o(\|t_j - x\|^n).$$

For any j with $\Phi_j(u) \neq 0$, we have u_j is in the support of Φ_j and hence $\|u - u_j\| \leqslant \frac{1}{2}\|u_j - t_j\|$. Consequently

$$\|u - x\| \geqslant \|u_j - x\| - \|u - u_j\| \geqslant \|u_j - t_j\| - \frac{1}{2}\|u_j - t_j\| = \frac{1}{2}\|u_j - t_j\|.$$

Thus

$$\begin{aligned}\|u_j - x\| &\leqslant \|u_j - u\| + \|u - x\| \\ &\leqslant \frac{1}{2}\|u_j - t_j\| + \|u - x\| \\ &\leqslant \frac{1}{2}\|u_j - x\| + \|u - x\|.\end{aligned}$$

Consequently $\|u_j - x\| \leqslant 2\|u - x\|$. Therefore

$$\|t_j - x\| \leqslant \|t_j - u_j\| + \|u_j - x\| \leqslant 2\|u - x\| + 2\|u - x\| = 4\|u - x\|$$

and hence the first sum in D is $o(\|u - x\|^n)$.

It remains to show that the second sum is $o(\|u-x\|^n)$. Throughout this argument it is always assumed that the index j is one for which $\Phi_j(u) \neq 0$. In this case u is in the support of Φ_j. Since $\|u - x\| \geqslant \frac{1}{2}\|u_j - t_j\|$, for $\|u - x\|$ sufficiently small there is a k_j so that $\dfrac{1}{B^2_{k_j+1}} < \|u - t_j\| \leqslant \dfrac{1}{B^2_{k_j}}$. Then for any α with $|\alpha| = n$

$$\begin{aligned}&\|(u-t_j)^\alpha(g_\alpha(u_j) - f_\alpha(x))\| \leqslant \|u - t_j\|^n \|g_\alpha(u_j) - f_\alpha(x)\| \\ &= \|u - t_j\|^n \|f_{\alpha,k_j}(t_j) - f_{\alpha,k_j}(x) + f_{\alpha,k_j}(x) - f_\alpha(x)\| \\ &\leqslant \|t_j - x\|\|u - t_j\|^{n-1}\|u - t_j\| \frac{\|f_{\alpha,k_j}(t_j) - f_{\alpha,k_j}(x)\|}{\|t_j - x\|} \\ &\quad + \|u - t_j\|^n \|f_{\alpha,k_j}(x) - f_\alpha(x)\| \\ &\leqslant 2\|u - x\| 3^{n-1}\|u - x\|^{n-1}\|u - t_j\| B_{k_j} + 3^n\|u - x\|^n \|f_{\alpha,k_j}(x) - f_\alpha(x)\| \\ &\leqslant 3^n\|u - x\|^n\Big(\|u - t_j\|B_{k_j} + \|f_{\alpha,k_j}(x) - f_\alpha(x)\|\Big) \\ &\leqslant 3^n\|u - x\|^n\Big(\frac{1}{B_{k_j}} + \|f_{\alpha,k_j}(x) - f_\alpha(x)\|\Big).\end{aligned}$$

As $u \to x$ then $t_j \to x$ as well. Thus $k_j \to \infty$ and hence $B_{k_j} \to \infty$. Consequently $f_{\alpha,k_j}(x) \to f_\alpha(x)$ and hence the second sum is $o(\|u - x\|^n)$. □

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