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THE STRUCTURE OF TRANSITIVE ORDERED PERMUTATION GROUPS

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Abstract. We give some necessary and sufficient conditions for transitive l-permutation groups to be 2-transitive. We also discuss primitive components and give necessary and sufficient conditions for transitive l-permutation groups to be normal-valued.

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1. Introduction

We continue the research ([1], [2], [3]) using stabilizer subgroups for transitive l-permutation groups. We will prove that a transitive l-permutation group is 2-transitive if and only if the stabilizer group G_α of a point α acts transitively on $\{w \mid w < \alpha\}$, if and only if for each $\gamma < \beta < \alpha$, there exists $g \in G_\alpha$ such that $\beta \leqslant \gamma g$. We will also discuss primitive components and will obtain that a transitive l-permutation group is normal-valued if and only if every primitive component is regular.

Let Ω be a chain and (G, Ω) an l-permutation group on Ω. If $\alpha G = \Omega$ for $\alpha \in \Omega$, then (G, Ω) is called transitive. Let Δ be a subset of Ω. The stabilizer of G on Δ, $G_\Delta = \{g \in G \mid \delta g = \delta \text{ for all } \delta \in \Delta\}$, G_α and $G_{(\Delta)} = \{g \in G \mid \Delta g = \Delta\}$ are prime subgroups of G ([2]). If Δ is a G_α-orbit, Δ is said to be positive if $\Delta > \{\alpha\}$ (negative if $\Delta < \{\alpha\}$). If Δ is an orbit of G_α and $|\Delta| > 1$, Δ is said to be a long orbit of G_α.

Let (G, Ω) be a transitive l-permutation group, $\alpha \in \Omega$, and let $(\mathcal{C}_k, \mathcal{C}^k)$ be a covering pair of convex congruences ([2]). Then $\alpha\mathcal{C}^k$ is a block of (G, Ω), and G induces an action on $\Omega/\mathcal{C}^k$. Let $\Omega_k = \alpha\mathcal{C}^k/\mathcal{C}_k$ be the chain of $\mathcal{C}_k$-classes within $\alpha\mathcal{C}^k$.

So the stabilizer $G_{(\alpha\mathcal{C}^k)}$ induces an order-permutation of Ω_k by G_k. (G_k, Ω_k) is called the k^{th} primitive component of (G, Ω) ([2]).

2. Transitivity

We first discuss the transitivity for l-permutation groups.

Lemma 1. *Let (G, Ω) be a transitive l-permutation group, and let Δ be a block. Then $\{\Delta g \mid g \in G\}$ is a partition of Ω, and the convex congruence associated with the partition is denoted by $\mathcal{C}_\Delta$. Furthermore each $g \in G$ induces an order-preserving permutation on the chain $\{\Delta g \mid g \in G\}$, i.e. $\Delta \to \Delta g$.*

Proof. The set $\{\Delta g \mid g \in G\}$ is a partition of Ω because of transitivity of G and the definition of the block ([2], Theorem 1.6.1). □

Theorem 2. *Let (G, Ω) be a transitive l-permutation group, and let Δ be a block. The following conditions are equivalent:*
(i) *$G_{(\Delta)}$ is a normal subgroup of G for each block Δ of (G, Ω).*
(ii) *$G_{(\Delta g)} = G_{(\Delta)}$ for every $g \in G$.*
(iii) *$G_{(\Delta)} = e$ where e is the identity.*
(iv) *$(G, \Omega/\mathcal{C}_\Delta)$ is regular, where $\mathcal{C}_\Delta$ is the congruence associated with the partition $\{\Delta g \mid g \in G\}$.*

Proof. The subgroup $G_{(\Delta)}$ is normal in G if and only if $G_{(\Delta)} = G_{(\Delta g)}$ by transitivity and the fundamental identity ([3]). But (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are obvious. Now we only prove that (iii) $\Rightarrow$ (iv). If $(\Delta f)h_i = \Delta g$, $i = 1, 2$, for any $f, g \in G$, then $\Delta f h_1 g^{-1} = \Delta f h_2 g^{-1} = \Delta$, i.e., $f h_i g^{-1} \in G_{(\Delta)} = e$. Hence $f h_i = g$, and $h_1 = h_2$. So $(G, \Omega/\mathcal{C}_{(\Delta)})$ is regular. □

Lemma 3. *Let (G, Ω) be a transitive l-permutation group. If the stabilizer subgroup G_α of a point α is transitive on $\{w \mid w < \alpha\}$, then the stabilizer subgroup G_β is also transitive on $\{w \mid w < \beta\}$ for every $\beta \in \Omega$.*

Proof. Let $\alpha = \beta f$ by the transitivity condition. If $\gamma, \delta < \beta$, then $\alpha f^{-1} > \gamma, \delta$ and $\alpha > \gamma f, \delta f$. There exists $g \in G_\alpha$ such that $(\delta f)g = \gamma f$ by hypothesis. So $\delta(f g f^{-1}) = \gamma$ and $f g f^{-1} \in G_\beta$ by the fundamental identity. Hence $\gamma \in \delta G_\beta$. □

Theorem 4. *Let (G, Ω) be a transitive l-permutation group. Then the following conditions are equivalent:*
(i) *(G, Ω) is 2-transitive.*
(ii) *The stabilizer subgroup G_α is transitive on $\{w \mid w < \alpha\}$.*

(iii) *For every* $\gamma < \beta < \alpha$, *there exists* $g \in G_\alpha$ *such that* $\beta \leqslant \gamma g$.

Proof. (i) $\Rightarrow$ (ii). If $\beta, \gamma \in \{w \mid w < \alpha\}$, i.e., $\beta < \alpha$, $\gamma < \alpha$, then there exists $g \in G$ such that $\beta = \gamma g$ and $\alpha g = \alpha$. So $\beta \leqslant \gamma g$ and $g \in G_\alpha$.

(ii) $\Rightarrow$ (iii). If $\gamma < \beta < \alpha$, i.e., $\gamma, \beta \in \{w \mid w < \alpha\}$, there is $g \in G_\alpha$ such that $\beta = \gamma g$, i.e. $\beta \leqslant \gamma g$.

(iii) $\Rightarrow$ (ii). If $\beta, \gamma \in \{w \mid w < \alpha\}$, let $\gamma < \beta < \alpha$. There is $g \in G_\alpha$ such that $\beta \leqslant \gamma g$. Let $\beta = \gamma f$ for some $f \in G$ by transitivity. Since $\alpha((f \vee e) \wedge g) = \alpha$, we have $(f \vee e) \wedge g \in G_\alpha$ and $\gamma((f \vee e) \wedge g) = \beta$.

(ii) $\Rightarrow$ (i). If $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, let $\alpha_2 h = \beta_2$ for some $h \in G$. Then $\alpha_1 h < \alpha_2 h = \beta_2$. For $\alpha_1 h$ and β_1, there exists $g \in G_{\beta_2}$ such that $(\alpha_1 h)g = \beta_1$. But we have also $\alpha_2 hg = \beta_2$. Hence G is 2-transitive. □

3. Primitivity

We now return to primitivity of l-permutation groups. Let α, β be distinct points of Ω. Let Δ be the union of blocks which contain α but not β. Then Δ is a block. Let Λ be the intersection of blocks containing both Δ and β. Then Λ is also a block, and Λ covers Δ in the chain of blocks containing α under inclusion. Let $\mathcal{C}^k$ and $\mathcal{C}_k$ be the convex congruences corresponding to Δ and Λ, respectively. Thus $(\mathcal{C}_k, \mathcal{C}^k)$ is a covering pair, and k is called the value $Val(\alpha, \beta)$ ([2]).

Theorem 5. *Let* (G, Ω) *be a transitive* l*-permutation group. Then the set* $K(G, \Omega) = \{(G_k, \Omega_k) \mid k \in K\}$ *of primitive components is a chain under inclusion. Moreover, every primitive component must be* 2*-transitive, regular or periodically primitive.*

Proof. Every block of (G, Ω) containing α is a chain because G is transitive, so $K(G, \Omega)$ is a chain where $(\mathcal{C}_k, \mathcal{C}^k) < (\mathcal{C}_{k'}, \mathcal{C}^{k'})$ if $\mathcal{C}^k \subsetneq \mathcal{C}^{k'}$ ([2], Theorem 3A). For the second part, every primitive component (G_k, Ω_k) is primitive ([2], Theorem 3E). Then G_k is 2-transitive, regular or periodically primitive ([2], Theorem 4.3.1). □

We have the following applications for the above Structure Theory.

Theorem 6. *Let* (G, Ω) *be a transitive* l*-permutation group. Then the following conditions are equivalent:*

(i) G *is normal-valued.*

(ii) $fg \leqslant g^2 f^2$ *for all* $f, g \in G^+$.

(iii) *All primitive components of* (G, Ω) *are regular.*

P r o o f. (i) $\Rightarrow$ (iii). Suppose that a primitive component (G_k, Ω_k) is not regular, then it must be 2-transitive or periodic. For every $\Delta \in \Omega_k$, $G_{(\Delta)}$ is not a normal subgroup of G_k. There is $g \in G$ such that $\Delta \neq \Delta g \in \alpha\mathcal{C}^k$, i.e., $g \notin G_{(\Delta)}$. Thus $G_{(\Delta)}$ is a value of G. By primitivity of $G_k, G_{(\Delta)}$ is a maximal prime subgroup of G_k. Hence G_k is a cover of $G_{(\Delta)}$. So G is not normal valued.

(iii) $\Rightarrow$ (ii). Suppose that $fg \nleqslant g^2f^2$ for some $f, g \in G^+$, then $\alpha fg > \alpha g^2 f^2$, where $\alpha \in \Omega$. Hence $\alpha fg > \alpha$. Let $k = Val(\alpha fg, \alpha)$. Then the primitive component (G_k, Ω_k) is regular. By primitivity of G_k, (G_k, Ω_k) is the regular representation of a subgroup of the set of real numbers $\mathcal{R}$. Let $\bar{f}$ and $\bar{g}$ be positive real translations induced respectively by f and g on $\alpha\mathcal{C}^k/\mathcal{C}_k$. Then we have $\Delta\bar{f}\bar{g} > \Delta\bar{g}^2\bar{f}^2$ where $\Delta = \alpha\mathcal{C}_k \in \alpha\mathcal{C}^k/\mathcal{C}_k$, a contradiction.

(ii) $\Rightarrow$ (i). By the Holland Representation Theorem, let $V(g)$ be a value. Then G is an l-subgroup of $A(\cup G/V(g))$ ([4], Theorem 5.4), and G is the transitive action on each individual space $G/V(g)$ for each $g \in G$. Let $V(g)^*$ be a cover of $V(g)$, and $G_{(V(g))} = V(g)$. Then $V(g)^*$ is the smallest prime subgroup containing $G_{(V(g))}$. Hence $G/V(g)$ has the smallest nontrivial convex congruence, and it has the smallest nonsingleton block of $(G, G/V(g))$. Suppose that a value $V(g)$ of G is not normal in its cover $V(g)^*$, and Δ is the smallest nonsingleton block containing the point $V(g)$. Since $V(g)$ is not normal in $G_{(\Delta)}$ and $G_{(\Delta)} = V(g)^*$, the primitive component $(G_{(\Delta)} \mid \Delta, \Delta)$ is not regular. So it must be 2-transitive or periodic, and it can not satisfy the identity "$fg \leqslant g^2f^2$ for all $f, g \in G^+$". Let z be periodic and $\Delta = (\alpha, \alpha z)$. Then $(G_{(\Delta)} \mid \Delta, \Delta)$ is 2-transitive ([2], Theorem 4.3.1). Then this identity must also fail in G. Suppose this identity holds for G. Then it must be true in $G_{(\Delta)}|\Delta$, which is an l-homomorphic image of an l-subgroup of an l-homomorphic image of G. $\square$

References

[1] *W. C. Holland*: Transitive lattice-ordered permutation groups. Math. Zeit. *87* (1965), 420–433; *MR31* (1966), # 2310.

[2] *A. M. W. Glass*: Ordered Permutation Groups. Cambridge University Press, 1981, pp. 76–116.

[3] *Z. T. Zhu and J. M. Huang*: Stability of l-permutation groups. J. of Nanjing Uni. Math. Biquarterly *11, No. 1* (1994), 18–21.

[4] *M. Anderson and T. Feil*: Lattice-Ordered Groups. D. Reidel Publishing Company, Dordrecht, Holland, 1988, pp. 29–31.

[5] *Z. T. Zhu and J. M. Huang*: Congruent pairs on a set. Chinese Quarterly Journal of Math. *9, No. 3* (1994), 37–41.

[6] *S. H. McClearly*: The structure of intransitive ordered permutation groups. Algebra Universalis *6* (1976), 229–255.

[7] *A. M. W. Glass*: Elementary types of automorphisms of linearly ordered sets—a survey. Algebra, Carbondale 1980 (R.K. Amayo, ed.). Springer, Lecture Notes No. 848, pp. 218–229.

[8] *S. H. McClearly*: The structure of ordered permutation groups applied to lattice-ordered groups. Notices Amer. Math. Soc., *21* (1974), February, # 712–714, PA336.

[9] *Z.T. Zhu and Q. Chen*: The universal mapping problems of the l-group category. Chinese Journal of Math. *23, No. 2* (1995), 131–140.

[10] *A. M. W. Glass and W. C. Holland (Eds)*: Lattice-Ordered Groups. Kluwer Academic Publishers, 1989, pp. 23–40.

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