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ON THE MIXED PROBLEM FOR HYPERBOLIC PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS OF THE FIRST ORDER

Tomasz Człapiński, Gdańsk

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Abstract. We consider the mixed problem for the hyperbolic partial differential-functional equation of the first order

$$D_x z(x,y) = f(x,y,z_{(x,y)},D_y z(x,y)),$$

where $z_{(x,y)}\colon [-\tau,0]\times[0,h]\to\mathbb{R}$ is a function defined by $z_{(x,y)}(t,s)=z(x+t,y+s)$, $(t,s)\in[-\tau,0]\times[0,h]$. Using the method of bicharacteristics and the method of successive approximations for a certain integral-functional system we prove, under suitable assumptions, a theorem of the local existence of generalized solutions of this problem.

Keywords: partial differential-functional equations, mixed problem, generalized solutions, local existence, bicharacteristics, successive approximations

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1. Introduction

If X,Y are any metric spaces then we denote by $C(X;Y)$ the class of all continuous functions from X to Y. Let $B=[-\tau,0]\times[0,h]$, where $h=(h_1,\ldots,h_n)\in\mathbb{R}^n_+$, $\tau\in\mathbb{R}_+$, ($\mathbb{R}_+=[0,+\infty)$). For a given function $z\colon [-\tau,\bar a]\times[-b,b+h]\to\mathbb{R}$, where $\bar a>0$, $b=(b_1,\ldots,b_n)$, $b_i>0$, $i=1,\ldots,n$, and a point $(x,y)=(x,y_1,\ldots,y_n)\in[0,a]\times[-b,b]$, we consider the function $z_{(x,y)}\colon B\to\mathbb{R}$ defined by

$$z_{(x,y)}(t,s)=z(x+t,y+s),\quad (t,s)\in B.$$

For any $a \in (0, \bar{a}]$ we define sets

$$E_0^* = [-\tau, 0] \times [-b, b+h], \qquad \partial_0 E_a = [0, a] \times [-b, b+h] \setminus [0, a] \times [-b, b),$$
$$E_a = [0, a] \times [-b, b], \qquad E_a^* = E_0^* \cup \partial_0 E_a \cup E_a.$$

For given functions $f\colon E_{\bar{a}} \times C(B; \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$, where $\bar{a} > 0$, and $\varphi\colon E_0^* \cup \partial_0 E_{\bar{a}} \to \mathbb{R}$, we consider the mixed problem

$$D_x z(x, y) = f(x, y, z_{(x,y)}, D_y z(x, y)), \tag{1}$$
$$z(x, y) = \varphi(x, y), \quad (x, y) \in E_0^* \cup \partial_0 E_{\bar{a}}, \tag{2}$$

where $D_y z = (D_{y_1} z, \ldots, D_{y_n} z)$.

We call $z\colon E_a^* \to \mathbb{R}$, where $0 < a \leqslant \bar{a}$, a solution of (1), (2) if

(i) $z \in C(E_a^*; \mathbb{R})$ and the derivative $D_y z(x, y)$ exists on E_a,

(ii) $z(\cdot, y)\colon [0, a] \to \mathbb{R}$ is absolutely continuous on $[0, a]$ for each $y \in [-b, b]$,

(iii) for any fixed $y \in [-b, b]$ equation (1) is satisfied for almost all $x \in [0, a]$, and condition (2) holds true for all $(x, y) \in E_0^* \cup \partial_0 E_a$.

In other words we wish to investigate the local (with respect to x) existence of generalized solutions of the problem (1), (2).

In this paper we deal with the problem in which the hereditary structure of the equation is based on the operator $(x, y) \mapsto z_{(x,y)}$. Note that in this setting f becomes a functional operator with respect to the third variable. Other settings are based on the use of abstract operators of the Volterra type or on the dependence of f on z with the assumption that f is of the Volterra type. Differential equations with a deviated argument and differential-integral equations are particular cases of (1).

There are various concepts of a solution concerning mixed problems for hyperbolic partial differential and differential-functional equations. Continuous solutions (satisfying integral systems arising from differential equations by integrating along bicharacteristics) of quasilinear systems were considered by Abolina and Myshkis [1] or Myshkis and Filimonov [17], [18]. Generalized (in the "almost everywhere" sense) solutions were investigated by Bassanini [2], Turo [20] and Kamont and Topolski [16] (see also [15]). Classical solutions in the functional setting were considered in [14].

In this paper we consider the mixed problem for the nonlinear differential-functional problem. Analogously to [7] we use the method of bicharacteristics together with the method of successive approximations for a certain integral-functional system. The method of bicharacteristics was introduced and developed in non-functional setting by Cinquini-Cibrario [11], [12] an Cinquini [10] for quasilinear as well as nonlinear problems. This method was adapted by Cesari [8], [9] and Bassanini [3], [4] for quasilinear systems in the second canonical form. Some extensions of Cesari's results to differential-functional systems were given in [5], [13], [19]. The results obtained in papers mentioned above by means of the method of bicharacteristics concern generalized solutions. Existence of generalized solutions to nonlinear differential-functional equations with the operator $z_{(x,y)}$ was proved by Brandi, Kamont and Salvadori [7]. An existence result for this equation was also obtained by Brandi and Ceppitelli [6] by means of the method of successive approximations.

2. Notation and assumptions

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with the norm $|\cdot|$ defined by $|y| = \max\limits_{1\leqslant i\leqslant n} |y_i|$. Let $C^{0,1}(B;\mathbb{R})$ be the set of all continuous functions $\omega\colon B\to\mathbb{R}$ of the variables $(t,s)=(t,s_1,\ldots,s_n)$ such that the derivative $D_s\omega=(D_{s_1}\omega,\ldots,D_{s_n}\omega)$ exists and is continuous on B. If $\|\cdot\|_0$ denotes the supremum norm in $C(B;\mathbb{R}^m)$ then the norm in $C^{0,1}(B;\mathbb{R})$ is defined by $\|\omega\|_1 = \|\omega\|_0 + \|D_s\omega\|_0$.

For any $\omega\in(B;\mathbb{R}^m)$ let

$$\|\omega\|_L = \sup\{|\omega(t,s)-\omega(\bar t,\bar s)|\cdot(|t-\bar t|+|s-\bar s|)^{-1}\colon\ (t,s),(\bar t,\bar s)\in B\}.$$

If we put $\|\omega\|_{0,L} = \|\omega\|_0 + \|\omega\|_L$, $\|\omega\|_{1,L} = \|\omega\|_1 + \|D_s\omega\|_L$, then we denote by $C^{0,i+L}(B;\mathbb{R})$, $i=0,1$, the space of all functions $\omega\in C^{0,i}(B;\mathbb{R})$ such that $\|\omega\|_{i,L} < +\infty$ with the norm $\|\cdot\|_{i,L}$.

Let $\Omega^{(0)} = E_{\bar a}\times C(B;\mathbb{R})\times\mathbb{R}^n$. Besides $\Omega^{(0)}$ we will consider the spaces $\Omega^{(1)} = E_{\bar a}\times C^{0,1}(B;\mathbb{R})\times\mathbb{R}^n$ and $\Omega^{(1,L)} = E_{\bar a}\times C^{0,1+L}(B;\mathbb{R})\times\mathbb{R}^n$.

Let $\|\cdot\|_{E_a}$, $\|\cdot\|_{E_a^*}$ denote the supremum norms in the spaces $C(E_a;\mathbb{R}^n)$, $C(E_a^*;\mathbb{R}^n)$, respectively.

Assumption H_1. Let $f\colon\ \Omega^{(0)}\to\mathbb{R}$ be a function of the variables (x,y,w,q) and let δ by any of these variables. Suppose that

1° the derivative $D_\delta f$ exists on $\Omega^{(1)}$, is measurable with respect to x and there is a nondecreasing function $\theta_1\colon\ \mathbb{R}_+\to\mathbb{R}_+$ such that

$$|D_\delta f(x,y,w,q)| \leqslant \theta_1(\|w\|_1) \quad \text{on} \quad \Omega^{(1)};$$

2° there is a nondecreasing function $\theta_2\colon\ \mathbb{R}_+\to\mathbb{R}_+$ such that for all $(x,y,w,q)\in\Omega^{(1,L)}$, $\bar y,\bar q\in\mathbb{R}^n$, $h\in C^{0,1}(B;\mathbb{R})$ we have

$$|D_\delta f(x,y,w,q) - D_\delta f(x,\bar y,w+h,\bar q)| \leqslant \theta_2(\|w\|_{1,L})[|y-\bar y| + \|h\|_1 + |q-\bar q|].$$

Remark 1. Note that if $\delta = w$ then for every $(x,y,w,q)\in\Omega^{(1)}$ the derivative $D_\delta f(x,y,w,q)$ is a continuous linear operator from $C^{0,1}(B;\mathbb{R})$ to $\mathbb{R}$. This means that in that case the norm of $D_\delta f(x,y,w,q)$ is a norm of a linear operator while if $\delta = y$ or $\delta = q$ it is a norm in the Euclidean space $\mathbb{R}^n$. These norms should be distinguished but for simplicity of notation we use the same symbol $|\cdot|$ in both cases.

Assumption H_2**.** Suppose that

1° $\varphi \in C(E_0^* \cup \partial_0 \cup \partial_0 E_{\bar a}; \mathbb{R})$, the derivative $D_y\varphi$ exists on $E_0^* \cup \partial_0 E_{\bar a}$;

2° there are constants $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathbb{R}_+$, such that

$$|\varphi(x,y)| \leqslant \Lambda_0, \quad |D_y\varphi(x,y)| \leqslant \Lambda_1 \quad \text{on} \quad E_0^* \cup \partial_0 E_{\bar a},$$
$$|D_y\varphi(x,y) - D_y\varphi(x,\bar y)| \leqslant \Lambda_2|y-\bar y| \quad \text{for } (x,y),(x,\bar y) \in E_0^* \cup \partial_0 E_{\bar a},$$

and furthermore

$$|\varphi(x,y)-\varphi(\bar x,y)| \leqslant \Lambda_1|x-\bar x|, \quad |D_y\varphi(x,y)-D_y\varphi(\bar x,y)| \leqslant \Lambda_2|x-\bar x| \quad \text{on } \partial_0 E_{\bar a} \cap E_{\bar a};$$

3° the derivative $D_x\varphi(x,y)$ exists on $\partial_0 E_{\bar a} \cap E_{\bar a}$ and the consistency condition

$$D_x\varphi(x,y) = f(x,y,\varphi_{(x,y)}, D_y\varphi(x,y)) \tag{3}$$

holds true on $\partial_0 E_{\bar a} \cap E_{\bar a}$.

Now, analogously to [7] we define two functional spaces such that the solution z of (1) will belong to the first space, while $D_y z$ to the other.

Let $Q = (Q_0, Q_1, Q_2)$, where $Q_i \in \mathbb{R}_+$, $Q_i \geqslant \Lambda_i$ for $i = 0,1,2$, and let $0 < a \leqslant \bar a$. If φ fulfils Assumption H_2 then we denote by $C^{0,1+L}_{\varphi,a}(Q)$ the set of all functions $z\colon E_a^* \to \mathbb{R}$ such that the derivative $D_y z$ exists on E_a^* and

(i) $z(x,y) = \varphi(x,y)$ on $E_0^* \cup \partial_0 E_a$;

(ii) $|z(x,y)| \leqslant Q_0$, $|D_y z(x,y)| \leqslant Q_1$ on E_a;

(iii) for $x, \bar x \in [0,a]$, $y, \bar y \in [-b,b]$, we have

$$|z(x,y) - z(\bar x,y)| \leqslant Q_1|x-\bar x|,$$
$$|D_y z(x,y) - D_y z(\bar x,\bar y)| \leqslant Q_2[|x-\bar x| + |y-\bar y|].$$

Let $P = (P_0, P_1)$, where $P_i \in \mathbb{R}_+$, $P_i \geqslant \Lambda_{i+1}$ for $i = 0,1$, and let $0 < a \leqslant \bar a$. If φ fulfils Assumption H_2 then we denote by $C^{0,L}_{D_y\varphi,a}(P)$ the set of all functions $u\colon E_a \to \mathbb{R}^n$ such that

(i) $u(x,y) = D_y\varphi(x,y)$ on $\partial_0 E_a \cap E_a$;

(ii) $|u(x,y)| \leqslant P_0$ on E_a;

(iii) for $x, \bar x \in [0,a]$, $y, \bar y \in [-b,b]$, we have

$$|u(x,y)| - u(\bar x,\bar y)| \leqslant P_1[|x-\bar x| + |y-\bar y|].$$

3. Bicharacteristics

For any $z \in C^{0,1+L}_{\varphi,a}(Q)$, $u \in C^{0,L}_{D_y\varphi,a}(P)$ we consider the Cauchy problem

$$\frac{\mathrm{d}\eta}{\mathrm{d}t}(t) = -D_q f(t, \eta(t), z_{(t,\eta(t))}, u(t, \eta(t))), \quad \eta(x) = y, \tag{4}$$

and we denote by $g[z,u](\cdot,x,y) = (g_1[z,u](\cdot,x,y), \ldots, g_n[z,u](\cdot,x,y))$ its solution, which exists if Assumption H_1 holds with $\delta = q$. Let $\lambda[z,u](x,y)$ be the left end of the maximal interval on which the solution $g[z,u](\cdot,x,y)$ is defined. If $D_{q_i} f(x,y,w,q) \geqslant 0$, $i = 1, \ldots, n$, on $\Omega^{(1)}$ then

$$(\lambda[z,u](x,y), g[z,u](\lambda[z,u](x,y),x,y)) \in (E_0^* \cup \partial_0 E_a) \cap E_a$$

and we may define the following two sets:

$$\begin{aligned} &E_{a0}[z,u] = \{(x,y) \in E_a \colon\ \lambda[z,u](x,y) = 0\},\\ &E_{ab}[z,u] = \{(x,y) \in E_a \colon\ g_i[z,u](\lambda[z,u](x,y),x,y) = b_i, \quad \text{for some } 1 \leqslant i \leqslant n\}. \end{aligned}$$

Remark 2. In the sequel we will write θ_i^*, $i = 0,1,2$, instead of $\theta_i\Bigl(\sum\limits_{j=0}^{i} Q_j\Bigr)$ for simplicity of notation.

Write

$$R_1 = 1 + Q_1 + Q_2 + P_1, \quad \Upsilon(t,x) = \exp\{R_1\theta_2^*|x-t|\}.$$

Lemma 1. *Suppose that $\varphi, \bar\varphi$ fulfil Assumption H_2 and that Assumption H_1 is satisfied for $\delta = q$. If $z \in C^{0,1+L}_{\varphi,a}(Q)$, $\bar z \in C^{0,1+L}_{\bar\varphi,a}(Q)$, $u \in C^{0,L}_{D_y\varphi,a}(P)$, $\bar u \in C^{0,L}_{D_y\bar\varphi,a}(P)$ are given functions and (x,y), $(\bar x,\bar y)$ are such that the intervals $K_1 = [\max\{\lambda[z,u](x,y), \lambda[z,u](\bar x,\bar y)\}, \min\{x,\bar x\}]$, $K_2 = [\max\{\lambda[z,u](x,y), \lambda[\bar z,\bar u](x,y)\}, x]$ are nonempty then we have the estimates*

$$|g[z,u](t,x,y) - g[z,u](t,\bar x,\bar y)| \leqslant \Upsilon(t,x)\{\theta_1^*|x-\bar x| + |y-\bar y|\} \quad \text{for } t \in K_1, \tag{5}$$

$$\begin{aligned} |g[z,u](t,x,y) - g[\bar z,\bar u](t,x,y)| \leqslant \Upsilon(t,x)\biggl|\int_x^t \theta_2^*\{\|z-\bar z\|_{E_\tau^*}\\ + \|D_y z - D_y\bar z\|_{E_\tau^*} + \|u-\bar u\|_{E_\tau}\}\,\mathrm{d}\tau\biggr| \quad \text{for } t \in K_2. \end{aligned} \tag{6}$$

P r o o f. Let $g = g[z,u]$ and $\bar g = g[\bar z,\bar u]$. If we transform (4) into an integral equation then by virtue of Assumption H_1 we have

$$\begin{aligned}
|g[z,u](t,x,y) - g[z,u](t,\bar x,\bar y)| &\leqslant |y-\bar y| + \left|\int_x^{\bar x} |D_q f(P[z,u](\tau,\bar x,\bar y))| d\tau\right| \\
&\quad + \left|\int_x^t |D_q f(P[z,u](\tau,x,y)) - D_q f(P[z,u](\tau,\bar x,\bar y))| \,\mathrm{d}\tau\right| \\
&\leqslant |y-\bar y| + \theta_1^*|x-\bar x| + \left|\int_x^t \theta_2^*\{|g[z,u](\tau,x,y) - g[z,u](\tau,\bar x,\bar y)|\right. \\
&\quad + \|z_{(\tau,g[z,u](\tau,x,y))} - z_{(\tau,g[z,u](\tau,\bar x,\bar y))}\|_1 \\
&\quad \left. + |u(\tau,g[z,u](\tau,x,y)) - u(\tau,g[z,u](\tau,\bar x,\bar y))|\}\,\mathrm{d}\tau\right| \\
&\leqslant |y-\bar y| + \theta_1^*|x-\bar x| + \left|\int_x^t \theta_2^* R_1 |g[z,u](\tau,x,y) - g[z,u](\tau,\bar x,\bar y)|\,\mathrm{d}\tau\right|
\end{aligned}$$

for $t \in K_1$, where

$$P[z,u](t,x,y) = (t, g[z,u](t,x,y), z_{(t,g[z,u](t,x,y))}, u(t,g[z,u](t,x,y))). \tag{7}$$

Thus (5) follows from the Gronwall lemma.

In the same way we get by Assumption H_1 the estimate

$$\begin{aligned}
&|g[z,u](t,x,y) - g[\bar z,\bar u](t,x,y)| \\
&\quad\leqslant \left|\int_x^t \theta_2^*\big\{\|z-\bar z\|_{E_\tau^*} + \|D_y z - D_y\bar z\|_{E_\tau^*} + \|u-\bar u\|_{E_\tau}\big\}\,\mathrm{d}\tau\right| \\
&\qquad + \left|\int_x^t \theta_2^* R_1 |g[z,u](\tau,x,y) - g[\bar z,\bar u](\tau,x,y)|\,\mathrm{d}\tau\right|
\end{aligned}$$

for $t \in K_2$. Now, again using the Gronwall lemma we get (6), which completes the proof of Lemma 1. $\square$

Lemma 2. *Suppose that $\varphi,\bar\varphi$ fulfil Assumption H_2 and that Assumption H_1 is satisfied for $\delta = q$. Furthermore, suppose that for every $p \in \mathbb{R}_+$ there is $\delta(p) > 0$ such that we have $D_{q_i} f(x,y,w,q) \geqslant \delta(p)$, $i = 1,\dots,n$, for $(x,y,w,q) \in \Omega^{(1)}$, $\|w\|_1 \leqslant p$. If $z \in C^{0,1+L}_{\varphi,a}(Q)$, $\bar z \in C^{0,1+L}_{\bar\varphi,a}(Q)$, $u \in C^{0,L}_{D_y\varphi,a}(P)$, $\bar u \in C^{0,L}_{D_y\bar\varphi,a}(P)$ are given functions then for all $(x,y),(\bar x,\bar y) \in E_a$ we have*

$$|\lambda[z,u](x,y) - \lambda[z,u](\bar x,\bar y)| \leqslant \frac{1}{\delta_0}\Upsilon(0,x)\{\theta_1^*|x-\bar x| + |y-\bar y|\}, \tag{8}$$

$$\begin{aligned}
|\lambda[z,u](x,y) - \lambda[\bar z,\bar u](x,y)| \leqslant \frac{1}{\delta_0}\Upsilon(0,x)\int_0^x \theta_2^*\big\{&\|z-\bar z\|_{E_\tau^*} + \|D_y z - D_y\bar z\|_{E_\tau^*} \\
&+ \|u-\bar u\|_{E_\tau^*}\big\}\,\mathrm{d}\tau,
\end{aligned} \tag{9}$$

where $\delta_0 = \delta(Q_0 + Q_1)$.

Proof. Let $g = g[z,u]$, $\lambda = \lambda[z,u]$, $\bar{g} = g[\bar{z},\bar{u}]$, $\bar{\lambda} = \lambda[\bar{z},\bar{u}]$. Since (8) is obviously satisfied if (x,y), $(\bar{x},\bar{y}) \in E_{a0}[z,u]$, without loss of generality we may assume that $\lambda(\bar{x},\bar{y}) \leqslant \lambda(x,y)$ and $(x,y) \in E_{ab}[z,u]$. Let $1 \leqslant i \leqslant n$ be such that $g_i(\lambda(x,y),x,y) = b_i$. Then we have

$$\begin{aligned} g_i(\lambda(x,y),x,y) &- g_i(\lambda(x,y),\bar{x},\bar{y}) \\ &\geqslant g_i(\lambda(\bar{x},\bar{y}),\bar{x},\bar{y}) - g_i(\lambda(x,y),\bar{x},\bar{y}) \\ &= \int_{\lambda(\bar{x},\bar{y})}^{\lambda(x,y)} D_{q_i} f(\tau, g(\tau,\bar{x},\bar{y}), z_{(\tau,g(\tau,\bar{x},\bar{y}))}, u(\tau, g(\tau,\bar{x},\bar{y})))\,\mathrm{d}\tau \\ &\geqslant \delta_0[\lambda(x,y) - \lambda(\bar{x},\bar{y})]. \end{aligned}$$

The above estimate together with (5) gives (8).

Analogously, since (9) is obviously satisfied if $(x,y) \in E_{a0}[z,u] \cap E_{a0}[\bar{z},\bar{u}]$ we may assume that $\bar{\lambda}(x,y) \leqslant \lambda(x,y)$ and $(x,y) \in E_{ab}[z,u]$. Then for $1 \leqslant i \leqslant n$ such that $g_i(\lambda(x,y),x,y) = b_i$ we have

$$\begin{aligned} g_i(\lambda(x,y),x,y) &- \bar{g}_i(\lambda(x,y),x,y) \\ &\geqslant \bar{g}_i(\bar{\lambda}(x,y),x,y) - \bar{g}_i(\lambda(x,y),x,y) \\ &= \int_{\bar{\lambda}(x,y)}^{\lambda(x,y)} D_{q_i} f(\tau, \bar{g}(\tau,x,y), \bar{z}_{(\tau,\bar{g}(\tau,x,y))}, \bar{u}(\tau, \bar{g}(\tau,x,y)))\,\mathrm{d}\tau \\ &\geqslant \delta_0[\lambda(x,y) - \bar{\lambda}(x,y)], \end{aligned}$$

which together with (6) gives (9). □

4. A certain system of integral-functional equations

Assumption H_3**.** Suppose that

1° Assumption H_1 is satisfied with $\delta = y, w, q$ and there is a nondecreasing function $\theta_0\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(x,y,w,q)| \leqslant \theta_0(\|w\|_0) \quad \text{on } \Omega^{(0)},$$

2° for every $p \in \mathbb{R}_+$ there is $\delta(p) > 0$ such that we have $D_{q_i} f(x,y,w,q) \geqslant \delta(p)$, $i = 1,\dots,n$, for $(x,y,w,q) \in \Omega^{(1)}$, $\|w\|_1 \leqslant p$.

If φ, f satisfy assumptions H_2, H_3 then for given $z \in C^{0,1+L}_{\varphi,a}(Q)$, $u \in C^{0,L}_{D_y\varphi,a}(P)$ we define the operators $T[z,u]$, $V_i[z,u]$, $i = 1,\ldots,n$, by

$$\begin{aligned}
T[z,u](x,y) = {}&\varphi(\lambda[z,u](x,y), g[z,u](\lambda[z,u](x,y),x,y))\\
&+ \int_{\lambda[z,u](x,y)}^{x} \Big[f(P[z,u](\tau,x,y))\\
&- \sum_{j=1}^{n} D_{q_j} f(P[z,u](\tau,x,y)) u_j(\tau, g[z,u](\tau,x,y))\Big]\,\mathrm{d}\tau,\\
V_i[z,u](x,y) = {}&D_{y_i}\varphi(\lambda[z,u](x,y), g[z,u](\lambda[z,u](x,y),x,y))\\
&+ \int_{\lambda[z,u](x,y)}^{x} \Big[D_{y_i} f(P[z,u](\tau,x,y))\\
&+ D_w f(P[z,u](\tau,x,y)) \circ (u_i)_{(\tau, g[z,u](\tau,x,y))}\Big]\,\mathrm{d}\tau
\end{aligned}$$

for $(x,y) \in E_a$, and

$$T[z,u](x,y) = \varphi(x,y),\ \ V_i[z,u](x,y) = D_{y_i}\varphi(x,y) \quad \text{for } (x,y) \in E_0^* \cup \partial_0 E_a,$$

where $g[z,u]$ is a solution of (4), $\lambda[z,u]$ is the left end of the maximal interval on which this solution is defied and $P[z,u]$ is given by (7). We will consider the system of integral-functional equations

$$\text{(10)} \qquad z = T[z,u], \quad u = V[z,u],$$

where $V[z,u] = (V_1[z,u],\ldots,V_n[z,u])$.

Remark 3. Integral-functional system (10) arises in the following way. We introduce an additional unknown function $u = D_y z$ in (1). Then we consider the linearization of (1) with respect to u which yields

$$\text{(11)} \qquad D_x z(x,y) = f(P) + \sum_{j=1}^{n} D_{q_j} f(P)(D_{y_j} z(x,y) - u_j(x,y)),$$

where $P = (x, y, z_{(x,y)}, u(x,y))$. Differentiating (1) with respect to y_i and substituting $u = D_y z$ we get

$$\text{(12)} \qquad \begin{aligned} D_x u_i(x,y) = {}&D_{y_i} f(P) + D_w f(P) \circ (u_i)_{(x,y)}\\ &+ \sum_{j=1}^{n} D_{q_j} f(P) D_{y_i} u_j(x,y), \quad i = 1,\ldots,n. \end{aligned}$$

Making use of (4) we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} z(\tau, g[z,u](\tau,x,y)) = D_x z(\tau, g[z,u](\tau,x,y)) - \sum_{j=1}^{n} D_{q_j} f(P[z,u](\tau,x,y)) D_{y_j} z(\tau, g[z,u](\tau,x,y)).$$

Substituting (11) in the above relation and integrating the resulting equation with respect to t on $[\lambda[z,u](x,y), x]$ we get the first of the equations in (10) on E_a^*. Repeating these considerations for (12) and taking into account that $z = \varphi$, $u = D_y\varphi$, on $E_0^* \cup \partial_0 E_a$ we get the second equation in (10).

Suppose that φ, f satisfy Assumptions H_2, H_3, respectively. Under these assumptions we prove by means of the method of successive approximations that the solution of (12) exists. We define a sequence $\{z^{(m)}, u^{(m)}\}$ in the following way:

1° Let $\widehat{\varphi}$ be any extension of φ onto the set E_a^* such that $\widehat{\varphi}$ satisfies conditions 1°, 2° of Assumption H_2 on E_a^*. We put

$$z^{(0)}(x,y) = \widehat{\varphi}(x,y), \quad u^{(0)}(x,y) = D_y\widehat{\varphi}(x,y), \tag{13}$$

and then $z^{(0)} \in C^{0,1+L}_{\varphi,a}(Q)$, $u^{(0)} \in C^{0,L}_{D_y\varphi,a}(P)$.

2° If $z^{(m)} \in C^{0,1+L}_{\varphi,a}(Q)$, $u^{(m)} \in C^{0,L}_{D_y\varphi,a}(P)$ are already defined functions then $u^{(m+1)}$ is a solution of the equation

$$u = V^{(m)}[z^{(m)}, u], \tag{14}$$

and

$$z^{(m+1)} = T[z^{(m)}, u^{(m+1)}], \tag{15}$$

where $V^{(m)}[z^{(m)}, u] = (V_1^{(m)}[z^{(m)}, u], \ldots, V_n^{(m)}[z^{(m)}, u])$ is defined by

$$\begin{aligned} V_i^{(m)}[z^{(m)}, u](x,y) = {} & D_{y_i}\varphi(\lambda[z^{(m)}, u](x,y), g[z^{(m)}, u](\lambda[z^{(m)}, u](x,y), x, y)) \\ & + \int_{\lambda[z^{(m)},u](x,y)}^{x} \big[D_{y_i} f(P[z^{(m)}, u](\tau,x,y)) \\ & + D_w f(P[z^{(m)}, u](\tau,x,y)) \circ (u_i^{(m)})_{(\tau, g[z^{(m)},u](\tau,x,y))} \big] \,\mathrm{d}\tau \end{aligned} \tag{16}$$

for $(x,y) \in E_a$, and

$$V_i^{(m)}[z^{(m)}, u](x,y) = D_{y_i}\varphi(x,y) \quad \text{for } (x,y) \in E_0^* \cup \partial_0 E_a.$$

Remark 4. Since the operators $V[z^{(m)},\cdot]$ and $V^{(m)}[z^{(m)},\cdot]$ are not identical we explain the way in which system (14) is obtained. If $z^{(m)} \in C^{0,1+L}_{\varphi,a}(Q)$, $u^{(m)} \in C^{0,L}_{D_y\varphi,a}(P)$ are known functions then replacing z with $z^{(m)}$ in system (12) we get

$$\begin{aligned} D_x u_i(x,y) = D_{y_i} f(P^{(m)}) &+ D_w f(P^{(m)}) \circ (D_{y_i} z^{(m)})_{(x,y)} \\ &+ \sum_{j=1}^{n} D_{q_j} f(P^{(m)}) D_{y_i} u_j(x,y), \quad i = 1,\dots,n, \end{aligned}$$

where $P^{(m)} = (x, y, z^{(m)}_{(x,y)}, u(x,y))$. If we assume that $D_y z^{(m)} = u^{(m)}$ (see Theorem 1), then by integrating the above system along the bicharacteristic $g[z^{(m)},u](\cdot,x,y)$ on the interval $[\lambda[z^{(m)},u](x,y),x]$ we get (14).

Write

$$\begin{aligned} \Gamma_0(x) &= \Lambda_1 + \theta_1^* S_1 x, \\ \widetilde{\Gamma}_0(x) &= \Lambda_1 \Upsilon(0,x)\Big[\frac{1}{\delta_0}(1+\theta_1^*)+1\Big]\theta_1^* + \Big[1 + \frac{1}{\delta_0}\Upsilon(0,x)\theta_1^*\Big](\theta_0^* + \theta_1^* P_0) \\ &\quad + \{\theta_1^* + \theta_2^* P_0\} R_1 \Upsilon(0,x) x, \\ \Gamma_1(x) &= \Lambda_2 \Upsilon(0,x)\Big[\frac{1}{\delta_0}(1+\theta_1^*)+1\Big] + S_1 + S_1\theta_1^*\frac{1}{\delta_0} \\ &\quad + \{\theta_2^* R_1 S_1 + \theta_1^* P_1\}\Upsilon(0,x)x, \\ G(x) &= \Lambda_2\Upsilon(0,x)\theta_2^*\Big[\frac{1}{\delta_0}(1+\theta_1^*)+1\Big] + \theta_1^* S_1 \frac{1}{\delta_0}\Upsilon(0,x)\theta_2^* \\ &\quad + [\theta_2^* R_1 S_1 + \theta_1^* P_1]\Upsilon(0,x)\theta_2^* x + \theta_2^* S_1, \end{aligned}$$

where

$$S_1 = 1 + P_0.$$

Assumption H_4**.** Suppose that we may choose constants $Q_i \in \mathbb{R}_+$, $Q_i > \Lambda_i$ for $i = 0,1,2$ such that $P_i = Q_{i+1}$ for $i = 0,1$, and that for sufficiently small $a \in (0,\bar{a}]$ we have the inequalities

$$\begin{aligned} &\Lambda_0 + [\theta_0^* + \theta_1^* P_0]a \leqslant Q_0, \quad \max\{\Gamma_0(a), \widetilde{\Gamma}(a)\} \leqslant Q_1, \\ &\max\{\Gamma_1(a), \theta_1^*\Gamma_1(a)\} \leqslant Q_2, \quad aG(a) < 1. \end{aligned}$$

5. The existence of the sequence of successive approximations

The problem of existence of the sequence $\{z^{(m)}, u^{(m)}\}$ is the main difficulty in our method. We prove that this sequence exists provided a, $0 < a \leqslant \bar{a}$, is sufficiently small.

Theorem 1. *If Assumptions* H_2–H_4 *are satisfied then for any* $m \in \mathbb{N}$ *we have*

(I_m) $z^{(m)}, u^{(m)}$ *are defined on* E_a^*, E_a, *respectively and we have* $z^{(m)} \in C^{0,1+L}_{\varphi,a}(Q)$, $u^{(m)} \in C^{0,L}_{D_y\varphi,a}(P)$;

(II_m) $D_y z^{(m)}(x,y) = u^{(m)}(x,y)$ *on* E_a.

Proof. We will prove (I_m) and (II_m) by induction. It follows from (15) that (I_0), (II_0) are satisfied. Suppose that conditions (I_m) and (II_m) hold true for a given $n \in \mathbb{N}$. We first prove that $u^{(m+1)}\colon E_a \to \mathbb{R}^n$ exists and $u^{(m+1)} \in C^{0,L}_{D_y\varphi,a}(P)$.

We claim that given $z^{(m)} \in C^{0,1+L}_{\varphi,a}(Q)$ the operator $V[z^{(m)}, \cdot]$ maps $C^{0,L}_{D_y\varphi,a}(P)$ into itself. For simplicity of notation we ignore the dependence of g, λ and P on $z^{(m)}$ and u. It follows from Assumptions H_2, H_3 and (5) that given $u \in C^{0,L}_{D_y\varphi,a}(P)$ then for all $(x,y), (\bar{x},\bar{y}) \in E_a$ we have the estimates

$$
\begin{aligned}
|V^{(m)}[z^{(m)},u](x,y)| &\leqslant \Lambda_1 + \int_{\lambda(x,y)}^{x} \theta_1^* S_1 \,\mathrm{d}\tau, \\
|V^{(m)}[z^{(m)},u](x,y) &- V^{(m)}[z^{(m)},u](\bar{x},\bar{y})| \\
&\leqslant \Lambda_2 \Upsilon(0,x)\Big\{[1+\theta_1^*]\frac{1}{\delta_0} + 1\Big\}\{\theta_1^*|x-\bar{x}| + |y-\bar{y}|\} \\
&\quad + \left|\int_{x}^{\bar{x}} \theta_1^* S_1 \,\mathrm{d}\tau\right| + \left|\int_{\lambda(x,y)}^{\lambda(\bar{x},\bar{y})} \theta_1^* S_1 d\tau\right| \\
&\quad + \{\theta_1^*|x-\bar{x}| + |y-\bar{y}|\} \cdot \int_{\lambda(x,y)}^{x} \{\theta_2^* R_1 S_1 + \theta_1^* P_1\}\Upsilon(\tau,x)\,\mathrm{d}\tau.
\end{aligned}
$$

Hence by Assumption H_4 we get

$$
\begin{aligned}
&|V^{(m)}[z^{(m)},u](x,y)| \leqslant P_0, \\
&|V^{(m)}[z^{(m)},u](x,y) - V^{(m)}[z^{(m)},u](\bar{x},\bar{y})| \leqslant P_1\big[|x-\bar{x}| + |y-\bar{y}|\big]
\end{aligned}
\tag{17}
$$

for $(x,y), (\bar{x},\bar{y}) \in E_a$. Since $V^{(m)}[z^{(m)},u] = D_y\varphi$ on $E_0^* \cup \partial_0 E_a$ it follows from (17) that $V^{(m)}[z^{(m)},\cdot]$ maps $C^{0,L}_{D_y\varphi,a}(P)$ into itself.

If $u, \bar{u} \in C^{0,L}_{D_y\varphi,a}(P)$, then analogously, by Assumptions H_2, H_3, (6), (9) and the relation $V^{(m)}[z^{(m)},u] = V^{(m)}[z^{(m)},\bar{u}] = D_y\varphi$ on $E_0^* \cup \partial_0 E_a$, we get

$$
\|V^{(m)}[z^{(m)},u] - V^{(m)}[z^{(m)},\bar{u}]\|_{E_a} \leqslant \int_0^a G(\tau)\|u-\bar{u}\|_{E_\tau}\,\mathrm{d}\tau.
$$

Thus Assumption H_4 yields that $V^{(m)}[z^{(m)},\cdot]$ is a contraction with the norm $\|\cdot\|_{E_a}$. By the Banach fixed point theorem there exists a unique solution $u \in C^{0,L}_{D_y\varphi,a}$ of (14) which is $u^{(m+1)}$.

Our next goal is to prove that $z^{(m+1)}$ given by (15) satisfies (II_{m+1}). For $x \in [0,a]$, $y,\bar{y} \in \mathbb{R}^n$ put

$$\Delta(x,y,\bar{y}) = z^{(m+1)}(x,y) - z^{(m+1)}(x,\bar{y}) - u^{(m+1)}(x,y)(y-\bar{y}).$$

By the Hadamard mean value theorem we have

$$\begin{aligned}
\Delta(x,y,\bar{y}) &= \varphi(\lambda(x,y), g(\lambda(x,y),x,y)) - \varphi(\lambda(x,\bar{y}), g(\lambda(x,\bar{y}),x,\bar{y}))\\
&\quad - D_y\varphi(\lambda(x,y), g(\lambda(x,y),x,y))(y-\bar{y})\\
&\quad + \int_{\lambda(x,y)}^{x}\int_0^1 D_yf(Q(s,\tau))[g(\tau,x,y)-g(\tau,x,\bar{y})]\,\mathrm{d}s\,\mathrm{d}\tau\\
&\quad + \int_{\lambda(x,y)}^{x}\int_0^1 D_wf(Q(s,\tau))\circ\big[z^{(m)}_{(\tau,g(\tau,x,y))} - z^{(m)}_{(\tau,g(\tau,x,\bar{y}))}\big]\,\mathrm{d}s\,\mathrm{d}\tau\\
&\quad + \int_{\lambda(x,y)}^{x}\int_0^1 D_qf(Q(s,\tau))\big[u^{(m+1)}(\tau,g(\tau,x,y)) - u^{(m+1)}(\tau,g(\tau,x,\bar{y}))\big]\,\mathrm{d}s\,\mathrm{d}\tau\\
&\quad - \int_{\lambda(x,y)}^{x}\big\{D_qf(\tau,x,y))u^{(m+1)}(\tau,g(\tau,x,y))\\
&\quad - D_qf(P(\tau,x,\bar{y}))u^{(m+1)}(\tau,g(\tau,x,\bar{y}))\big\}\,\mathrm{d}\tau\\
&\quad + \int_{\lambda(x,y)}^{\lambda(x,\bar{y})}\big\{f(P(\tau,x,\bar{y})) - D_qf(P(\tau,x,\bar{y}))u^{(m+1)}(\tau,g(\tau,x,\bar{y})\big\}\,\mathrm{d}\tau\\
&\quad - \int_{\lambda(x,y)}^{x}\big\{D_yf(P(\tau,x,y)) + D_wf(P(\tau,x,y))\circ u^{(m)}_{(\tau,g(\tau,x,y))}\big\}\,\mathrm{d}\tau(y-\bar{y}),
\end{aligned}$$

where $Q(s,\tau) = sP(\tau,x,y) + (1-s)P(\tau,x,\bar{y})$. Let us define

$$\begin{aligned}
\Delta_0(x,y,\bar{y}) &= \varphi(\lambda(x,y), g(\lambda(x,y),x,y)) - \varphi(\lambda(x,\bar{y}), g(\lambda(x,\bar{y}),x,\bar{y}))\\
&\quad - D_x\varphi(\lambda(x,y), g(\lambda(x,y),x,y))[\lambda(x,y)-\lambda(x,\bar{y})]\\
&\quad - D_y\varphi(\lambda(x,y), g(\lambda(x,y),x,y))[g(\lambda(x,y),x,y) - g(\lambda(x,\bar{y}),x,\bar{y})],\\
\Delta_1(x,y,\bar{y}) &= \int_{\lambda(x,y)}^{x}\int_0^1 \big[D_yf(Q(s,\tau)) - D_yf(P(\tau,x,y))\big]\\
&\quad \times [g(\tau,x,y)-g(\tau,x,\bar{y})]\,\mathrm{d}s\,\mathrm{d}\tau,\\
\Delta_2(x,y,\bar{y}) &= \int_{\lambda(x,y)}^{x}\int_0^1 \big[D_wf(Q(s,\tau)) - D_wf(P(\tau,x,y))\big]\\
&\quad \circ\big[z^{(m)}_{(\tau,g(\tau,x,y))} - z^{(m)}_{(\tau,g(\tau,x,\bar{y}))}\big]\,\mathrm{d}s\,\mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}
\Delta_3(x,y,\bar{y}) = &\int_{\lambda(x,y)}^{x}\int_0^1 \big[D_q f(Q(s,\tau)) - D_q f(P(\tau,x,\bar{y}))\big]\\
&\times \big[u^{(m+1)}(\tau,g(\tau,x,y)) - u^{(m+1)}(\tau,g(\tau,x,\bar{y}))\big]\,\mathrm{d}s\,\mathrm{d}\tau,\\
\Delta_4(x,y,\bar{y}) = &\int_{\lambda(x,y)}^{x} D_w f(P(\tau,x,y)) \circ \big[z^{(m)}_{(\tau,g(\tau,x,y))} - z^{(m)}_{(\tau,g(\tau,x,\bar{y}))}\\
&- u^{(m)}_{(\tau,g(\tau,x,y))}[g(\tau,x,y) - g(\tau,x,\bar{y})]\big]\,\mathrm{d}\tau,\\
\Delta_5(x,y,\bar{y}) = &\,[\lambda(x,y) - \lambda(x,\bar{y})]\cdot D_x\varphi(\lambda(x,y),g(\lambda(x,y),x,y))\\
&- \int_{\lambda(x,\bar{y})}^{\lambda(x,y)} f(P(\tau,x,\bar{y}))\,\mathrm{d}\tau,\\
\Delta_6(x,y,\bar{y}) = &\,[g(\lambda(x,y),x,\bar{y}) - g(\lambda(x,\bar{y}),x,\bar{y})]\cdot D_y\varphi(\lambda(x,y),g(\lambda(x,y),x,y))\\
&+ \int_{\lambda(x,\bar{y})}^{\lambda(x,y)} D_q f(P(\tau,x,\bar{y}))u^{(m+1)}(\tau,g(\tau,x,\bar{y}))\,\mathrm{d}\tau,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\Delta}_0(x,y,\bar{y}) = &\,D_y\varphi(\lambda(x,y),g(\lambda(x,y),x,y))[g(\lambda(x,y),x,y)\\
&- g(\lambda(x,y),x,\bar{y}) - (y-\bar{y})],\\
\widetilde{\Delta}_1(x,y,\bar{y}) = &\int_{\lambda(x,y)}^{x} D_y f(P(\tau,x,y))[g(\tau,x,y) - g(\tau,x,\bar{y}) - (y-\bar{y})]\,\mathrm{d}\tau\\
&+ \int_{\lambda(x,y)}^{x} D_w f(P(\tau,x,y))\\
&\circ u^{(m)}_{(\tau,g(\tau,x,y))}[g(\tau,x,y) - g(\tau,x,\bar{y}) - (y-\bar{y})]\,\mathrm{d}\tau,\\
\widetilde{\Delta}_2(x,y,\bar{y}) = &-\int_{\lambda(x,y)}^{x} [D_q f(P(\tau,x,y)) - D_q f(P(\tau,x,\bar{y}))]u^{(m+1)}(\tau,g(\tau,x,y))\,\mathrm{d}\tau.
\end{aligned}$$

With the above definitions we have

$$\Delta(x,y,\bar{y}) = \sum_{i=0}^{6}\Delta_i(x,y,\bar{y}) + \sum_{i=0}^{2}\widetilde{\Delta}_i(x,y,\bar{y}). \tag{18}$$

Since $g(\cdot,x,y)$ is a solution of (4) we see that

$$g(\tau,x,y) - g(\tau,x,\bar{y}) - (y-\bar{y}) = \int_{\tau}^{x} [D_q f(P(\xi,x,y)) - D_q f(P(\xi,x,\bar{y}))]\,\mathrm{d}\xi.$$

Substituting the above relation in $\widetilde{\Delta}_1$ and in $\widetilde{\Delta}_0$ with $\tau=0$ and changing the order of integrals where necessary we get

$$\begin{aligned}\sum_{i=0}^{2}\widetilde{\Delta}(x,y,\bar{y}) = &\int_{\lambda(x,y)}^{x}\big[D_q f(P(\tau,x,y)) - D_q f(P(\tau,x,\bar{y}))\big]\Big[D_y\varphi(0,g(0,x,y))\\ &+\int_{\lambda(x,y)}^{\tau} D_y f(P(\xi,x,y))\,\mathrm{d}\xi\\ &+\int_{\lambda(x,y)}^{\tau} D_w f(P(\xi,x,y))\circ u^{(m)}_{(\xi,g(\xi,x,y))}\,\mathrm{d}\xi - u^{(m+1)}(\tau,g(\tau,x,y))\Big]\,\mathrm{d}\tau\\ =&\int_{\lambda(x,y)}^{x}\big[D_q f(P(\tau,x,y)) - D_q f(P(\tau,x,\bar{y}))\big]\\ &\times\big[V^{(m)}[z^{(m)},u^{(m+1)}](\tau,g(\tau,x,y)) - u^{(m+1)}(\tau,g(\tau,x,y))\big]\,\mathrm{d}\tau = 0,\end{aligned}$$

from which and from (18) we get $\Delta(x,y,\bar{y}) = \sum\limits_{i=0}^{6}\Delta_i(x,y,\bar{y})$. In the above transformations we have used the group property

$$g(\xi,\tau,g(\tau,x,y)) = g(\xi,x,y) \quad \text{for} \quad (x,y)\in E_a,\tau,\xi\in[0,a].$$

Assumptions H_2, H_3, (5) and the existence of derivatives $D_y\varphi$, $D_y z^{(m)} = u^{(m)}$ yield that for $x\in[0,a]$, $i=0,4$, we have

$$\frac{1}{|y-\bar{y}|}\Delta_i(x,y,\bar{y}) \to 0 \quad \text{if } |y-\bar{y}|\to 0. \tag{19}$$

From Assumption H_3 and (5) we get the existence of some constants C_i, $i=1,2,3$, such that

$$|\Delta_i(x,y,\bar{y})| \leqslant C_i|y-\bar{y}|^2, \quad x\in[0,a],\; y,\bar{y}\in[-b,b],\; i=1,2,3.$$

Writing Δ_5, Δ_6 in the form

$$\begin{aligned}\Delta_5(x,y,\bar{y}) &= \int_{\lambda(x,\bar{y})}^{\lambda(x,y)}\big[D_x\varphi(\lambda(x,y),g(\lambda(x,y),x,y)) - f(P(\tau,x,\bar{y}))\big]\,\mathrm{d}\tau,\\ \Delta_6(x,y,\bar{y}) &= \int_{\lambda(x,\bar{y})}^{\lambda(x,y)} D_q f(P(\tau,x,\bar{y}))\big[u^{(m+1)}(\tau,g(\tau,x,\bar{y}))\\ &\quad - D_y\varphi(\lambda(x,y),g(\lambda(x,y),x,y))\big]\,\mathrm{d}\tau\end{aligned}$$

and making use of the consistency condition (3) and the relation $u^{(m+1)} = D_y\varphi$ on $\partial_0 E_a\cap E_a$ we get estimates of the same type for $i=5,6$. This means that (19) holds true also for $i=1,2,3,5,6$, which completes the proof of (II_{m+1}).

Finally, we prove that $z^{(m+1)}$ defined by (15) belongs to the class $C^{0,1+L}_{\varphi,a}(Q)$. Since $D_y z^{(m+1)} = u^{(m+1)}$ it follows from (17) and from Assumption H_4 that

$$\begin{aligned}
&|D_y z^{(m+1)}(x,y)| \leqslant Q_1,\\
&|D_y z^{(m+1)}(x,y) - D_y z^{(m+1)}(\bar{x},\bar{y})| \leqslant Q_2[|x-\bar{x}| + |y-\bar{y}|]
\end{aligned}$$

for $(x,y),(\bar{x},\bar{y}) \in E_a$. By Assumptions H_2–H_4 we easily get

$$|z^{(m+1)}(x,y)| \leqslant Q_0, \quad |z^{(m+1)}(x,y) - z^{(m+1)}(\bar{x},y)| \leqslant Q_1|x-\bar{x}|$$

for $(x,y),(\bar{x},y) \in E_a$. This together with the relation $z^{(m+1)} = \varphi$ on $E_0^* \cup \partial_0 E_a$ gives $z^{(m+1)} \in C^{0,1+L}_{\varphi,a}(Q)$, which completes the proof of (I_{m+1}). Thus Theorem 1 follows by induction. □

6. The main result

Write

$$H^*(t) = H(t) + H(t)\exp\left\{\int_0^t G(\xi)\,\mathrm{d}\xi\right\}\int_0^t G(\xi)\,\mathrm{d}\xi,$$

where

$$\begin{aligned}
H(t) = \Lambda_1\Upsilon(0,t)\theta_2^*\Big[\frac{1}{\delta_0}(1+\theta_1^*)+1\Big] &+ \theta_1^* S_1\frac{1}{\delta_0}\Upsilon(0,t)\theta_2^*\\
&+ [\theta_2^* R_1 P_0 + \theta_1^* R_1]\Upsilon(0,t)\theta_2^* t + \theta_1^* + \theta_2^* P_0.
\end{aligned}$$

Theorem 2. *If Assumptions* H_2–H_4 *are satisfied then the sequences* $\{z^{(m)}\}$, $\{u^{(m)}\}$ *are uniformly convergent on* E_a.

P r o o f. For any $t \in [0,a]$ and $m \in \mathbb{N}$ we put

$$\begin{aligned}
Z^{(m)}(t) &= \sup\big\{|z^{(m)}(x,y) - z^{(m-1)}(x,y)|\colon\ (x,y) \in E_t\big\},\\
U^{(m)}(t) &= \sup\big\{|u^{(m)}(x,y) - u^{(m-1)}(x,y)|\colon\ (x,y) \in E_t\big\}.
\end{aligned}$$

Using the same technique as in the proof of Theorem 1 we get by Assumptions H_2, H_3 and (6) for any $x \in [0,a]$ and $m \in \mathbb{N}$ the estimate

$$U^{(m+1)}(x) \leqslant \int_0^x G(\tau)U^{(m+1)}(\tau)\,\mathrm{d}\tau + \int_0^x G(\tau)\big[Z^{(m)}(\tau) + U^{(m)}(\tau)\big]\,\mathrm{d}\tau.$$

Making use of the Gronwall lemma we have

$$U^{(m+1)}(x) \leqslant \exp\left\{\int_0^x G(\tau)\,\mathrm{d}\tau\right\}\int_0^x G(\tau)\big[Z^{(m)}(\tau) + U^{(m)}(\tau)\big]\,\mathrm{d}\tau. \tag{20}$$

By Assumptions H_2, H_3, (10) and (20) we get the estimate

$$Z^{(m+1)}(x) \leqslant \int_0^x H^*(\tau)\big[Z^{(m)}(\tau) + U^{(m)}(\tau)\big]\,\mathrm{d}\tau, \quad x \in [0,a]. \tag{21}$$

Thus if we take

$$M_a = \exp\Big\{\int_0^a G(\xi)\,\mathrm{d}\xi\Big\} G(a) + H^*(a),$$

then using (20), (21) for any $x \in [0,a]$ we have

$$Z^{(m+1)}(x) + U^{(m+1)}(x) \leqslant M_a \int_0^x \big[Z^{(m)}(\tau) + U^{(m)}(\tau)\big]\,\mathrm{d}\tau.$$

Now, by induction it is easy to get

$$Z^{(m)}(x) + U^{(m)}(x) \leqslant \frac{M_a^{m-1}x^{m-1}}{(m-1)!}\big[Z^{(1)}(a) + U^{(1)}(a)\big], \quad x \in [0,a],$$

and consequently

$$\sum_{i=k}^{m}[Z^{(i)}(a) + U^{(i)}(a)] \leqslant [Z^{(1)}(a) + U^{(1)}(a)] \sum_{i=k-1}^{m-1} \frac{M_a^i a^i}{i!}. \tag{22}$$

Since the series $\sum\limits_{i=1}^{\infty} \frac{M_a^i a^i}{i!}$ is convergent it follows from (22) that the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ satisfy the uniform Cauchy condition on E_a, which means that they are uniformly convergent on E_a. This completes the proof of Theorem 2. $\square$

Theorem 3. *If Assumptions* H_2–H_4 *are satisfied then there is a solution of the problem* (1), (2).

P r o o f. It follows from Theorem 2 that there exist functions $\bar z, \bar u$ such that $\{z^{(m)}\}$, $\{u^{(m)}\}$ are uniformly convergent on E_a to $\bar z$, $\bar u$, respectively. Furthermore, $D_y \bar z$ exists on E_a and $D_y \bar z = \bar u$. We prove that $\bar z$ is a solution of (1).

From (12) it follows that for any $(x,y) \in E_{a0}[\bar z, D_y \bar z]$ we have

$$\begin{aligned} \bar z(x,y) = \varphi(0, \bar g(0,x,y)) + \int_0^x \Big[& f(P[\bar z, D_y \bar z](\tau,x,y)) \\ & - \sum_{j=1}^{n} D_{q_j} f(P[\bar z, D_y \bar z](\tau,x,y)) D_{y_j} \bar z(\tau,x,y)\Big]\,\mathrm{d}\tau, \end{aligned} \tag{23}$$

where $\bar g = g[\bar z, D_y \bar z]$.

For a fixed x we define the transformation $y \mapsto \bar{g}(0,x,y) = \xi$. Then by the group property $\bar{g}(t,x,y) = \bar{g}(t,0,\xi)$ and by (23) we get

$$\bar{z}(x,\bar{g}(x,0,\xi)) = \varphi(0,\xi) + \int_0^x \Big[f(\tau,\bar{g}(\tau,0,\xi),\bar{z}_{(\tau,\bar{g}(\tau,0,\xi))},D_y\bar{z}(\tau,\bar{g}(\tau,0,\xi))) \\ - \sum_{j=1}^n D_{q_j} f(\tau,\bar{g}(\tau,0,\xi),\bar{z}_{(\tau,\bar{g}(\tau,0,\xi))},D_y\bar{z}(\tau,\bar{g}(\tau,0,\xi)))D_{y_j}\bar{z}(\tau,\bar{g}(\tau,0,\xi))\Big]\,\mathrm{d}\tau.$$

Differentiating the above relation with respect of x and making use of the reverse transformation $\xi \mapsto \bar{g}(x,0,\xi) = y$, we see that $\bar{z}$ satisfies (1) for almost all x with fixed y on $E_{a0}[\bar{z},D_y\bar{z}]$.

Analogously for any $(x,y) \in E_{ab}[\bar{z},D_y\bar{z}]$ we have

$$\bar{z}(x,y) = \varphi(0,\bar{g}(0,x,y)) + \int_{\bar{\lambda}(x,y)}^x \Big[f(P[\bar{z},D_y\bar{z}](\tau,x,y)) \\ - \sum_{j=1}^n D_{q_j} f(P[\bar{z},D_y\bar{z}](\tau,x,y))D_{y_j}\bar{z}(\tau,x,y)\Big]\,\mathrm{d}\tau, \tag{24}$$

where $\bar{\lambda} = \lambda[\bar{z},D_y\bar{z}]$. For simplicity of notation suppose that $\bar{g}_i(\bar{\lambda}(x,y),x,y) = b_i$ for $i = n$ and write $\xi' = (\xi_1,\ldots,\xi_{n-1})$, $\bar{g}'_i(\bar{g}_1,\ldots,\bar{g}_{n-1})$. For a fixed x we define the transformation $y \mapsto (\bar{g}'(\bar{\lambda}(x,y),x,y),\bar{\lambda}(x,y)) = (\xi',\eta)$. Then by (24) and the group property we get

$$\bar{z}(x,\bar{g}(x,\eta,\xi',b_n)) = \varphi(\eta,\xi',b_n) \\ + \int_\eta^x \Big[f(\tau,\bar{g}(\tau,\eta,\xi',b_n),\bar{z}_{(\tau,\bar{g}(\tau,\eta,\xi',b_n))},D_y\bar{z}(\tau,\bar{g}(\tau,\eta,\xi',b_n))) \\ - \sum_{j=1}^n D_{q_j} f(\tau,\bar{g}(\tau,\eta,\xi',b_n),\bar{z}_{(\tau,\bar{g}(\tau,\eta,\xi',b_n))},D_y\bar{z}(\tau,\bar{z}(\tau,\bar{g}(\tau,\eta,\xi',b_n))) \\ \times D_{y_j}\bar{z}(\tau,\bar{g}(\tau,\eta,\xi',b_n))\Big]\,\mathrm{d}\tau.$$

Differentiating the above relation with respect to x and making use of the reverse transformation $(\xi',\eta) \mapsto \bar{g}(x,\eta,\xi',b_n) = y$, we see that $\bar{z}$ satisfies (1) for almost all x with fixed y also on $E_{ab}[\bar{z},D_y\bar{z}]$. Since obviously $\bar{z}$ fulfils condition (2), the proof of Theorem 3 is complete. □

Remark 5. If in Theorem 3 we assume that f is continuous then we get existence of classical solutions of problem (1), (2).

Remark 6. The existence results of our paper can be extended to weak coupled differential-functional systems

$$D_x z_i(x,y) = f_i(x,y,z_{(x,y)}, D_y z_i(x,y)), \quad i=1,\dots,k,$$
$$z_i(x,y) = \varphi_i(x,y), \quad (x,y) \in E_0^* \cup \partial_0 E_{\bar a},\ i=1,\dots,k,$$

where $z=(z_1,\dots,z_k)$, with given functions $f_i\colon E_{\bar a} \times C(B;\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}$ and $\varphi_i\colon E_0^* \cup \partial_0 E_{\bar a} \to \mathbb{R}$.

Now, we show some examples of differential-functional equations which are particular cases of (1).

Example 1. Given $\widehat{f}\colon E_{\bar a} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ let us consider the differential equation with a deviated argument

$$D_x z(x,y) = \widehat{f}(x,y,z(\alpha(x),\beta(x,y)), D_y z(x,y)), \tag{25}$$

where $\alpha\colon [0,\bar a] \to \mathbb{R}$, $\beta\colon E_{\bar a} \to [-b,b]$, and $\alpha(x) \leqslant x$ for $x \in [0,\bar a]$. We define a function f by

$$f(x,y,w,q) = \widehat{f}(x,y,w(\alpha(x)-x,\beta(x,y)-y),q)$$

for $(x,y,w,q) \in E_{\bar a} \times C(B;\mathbb{R}) \times \mathbb{R}^n$. If $(\alpha(x)-x,\beta(x,y)-y) \in B$ for $(x,y) \in E_{\bar a}$ then (25) is a particular case of (1) under natural assumptions on $\alpha,\beta,\widehat{f}$.

Example 2. With $\widehat{f}$ as in the previous example consider the differential-integral equation

$$D_x z(x,y) = \widehat{f}\Big(x,y,\int_B z(x+t,y+s)\,\mathrm{d}t\,\mathrm{d}s, D_y z(x,y)\Big). \tag{26}$$

If we define a function f by

$$f(x,y,w,q) = \widehat{f}\Big(x,y,\int_B w(t,s)\,\mathrm{d}t\,\mathrm{d}s, q\Big)$$

for $(x,y,w,q) \in E_{\bar a} \times C(B;\mathbb{R}) \times \mathbb{R}^n$, then it is easy to formulate assumptions on $\widehat{f}$ in order to get the existence theorem for (26) as a particular case of (1).

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Author's address: Institute of Mathematics, University of Gdańsk, Wit Stwosz Str. 57, 80-952 Gdańsk, Poland.