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ON THE TAUBERIAN CONSTANT IN THE IKEHARA THEOREM

Jiří Čížek,[1] Plzeň

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Let p denote any prime. Let $\Lambda(n) = \log p$ for $n = p^r$, $r \in \mathbb{N}$, $\Lambda(n) = 0$ otherwise. Then the prime number theorem, which reads

$$\pi(x) = \sum_{p \leqslant x} 1 = \int_2^x \frac{\mathrm{d}u}{\log u} + o\left(\frac{x}{\log x}\right), \quad x \to +\infty,$$

is equivalent to $\psi(x) = \sum\limits_{n \leqslant x} \Lambda(n) = x + o(x)$, $x \to +\infty$. The last relation follows from the Ikehara theorem: Let $A(x)$ be a nonnegative nondecreasing function defined for $x \in \langle 0; +\infty)$ and let the integral $f(s) = \int_0^\infty A(x)\mathrm{e}^{-xs}\,\mathrm{d}x$, $s = \sigma + \mathrm{i}t$, converge for $\sigma > 1$. Let $f(s)$ be analytic for $\sigma \geqslant 1$, except for a simple pole at $s = 1$ with residue 1. Then $\lim\limits_{x \to +\infty} \mathrm{e}^{-x}A(x) = 1$ (cf. [1], p. 124).

Of course, it suffices to put $A(x) = \psi(\mathrm{e}^x), x \geqslant 0$. Since

$$\int_0^\infty A(x)\mathrm{e}^{xs}\,\mathrm{d}x = -\frac{\xi'(s)}{s\xi(s)} = \frac{1}{s-1} + h(s) \qquad \text{for } \sigma > 1,$$

where $h(s)$ is analytic for $\sigma \geqslant 1$, we can use the Ikehara theorem to obtain $\lim\limits_{x \to +\infty} \mathrm{e}^{-x}A(x) = \lim\limits_{x \to +\infty} \mathrm{e}^{-x}\psi(\mathrm{e}^x) = \lim\limits_{x \to +\infty} \frac{\psi(x)}{x} = 1$.

To estimate the remainder term $\pi(x) - \int_2^x \frac{\mathrm{d}u}{\log u}$ we need a more sophisticated way: Let functions $\omega(x)$ and $\frac{1}{6}\log x - \omega(x)$ be both positive and increasing in $\langle 3; +\infty)$. Then the relation $\psi_1(x) = \sum\limits_{n \leqslant x} \Lambda(n) \log \frac{x}{n} = x + O(x \exp(-2\omega(x)))$, $x \geqslant 3$, implies $\pi(x) = \int_2^x \frac{\mathrm{d}u}{\log u} + O(x \exp(-\omega(x)))$, $x \geqslant 3$ (see [5]). We know that

$$\begin{aligned} h_1(s) = &-\frac{\xi'(s)}{s^2\xi(s)} - \frac{1}{s-1} = \int_0^\infty \psi_1(\mathrm{e}^x)\mathrm{e}^{-xs}\,\mathrm{d}x - \frac{1}{s-1} \\ &= \int_0^\infty (\mathrm{e}^{-x}\psi_1(\mathrm{e}^x) - 1)\mathrm{e}^{-x(s-1)}\,\mathrm{d}x \end{aligned}$$

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is analytic for $\sigma \geqslant 1$. However, we now need a better estimate of the remainder term $\mathrm{e}^{-x}\psi_1(\mathrm{e}^x) - 1$ than $o(1)$, $x \to +\infty$, which the Ikehara theorem yields. Theorem 2 of [2] is such an Ikehara theorem with the remainder term. If moreover the function $g(t) = f(1+\mathrm{i}t) - \frac{1}{\mathrm{i}t}$, $t \in \mathbb{R}$, satisfies $g^{(n)} \in L^1(\mathbb{R})$, then $\mathrm{e}^{-x}A(x) = 1 + O(x^{-n})$, $x \to +\infty$. This last theorem yields $\psi_1(x) = x + O(x \log^{-n} x)$, $x \to +\infty$. Theorem 0.1 of the present paper allows us to show how the constant in O in this relation depends on $n \in \mathbb{N}$.

We denote $\|f\|_\infty = \sup\limits_{x\in\mathbb{R}} |f(x)|$ for any function $f\colon \mathbb{R} \to \mathbb{C}$ and $\|f\|_1 = \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t$. Further, let $\hat{f}(x) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}t$ be the Fourier transform of a function f for $f \in L^1(\mathbb{R})$. Let $M_0 = M_{-1} = 1$ and let $\{M_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that

$$C = \sum_{k=0}^{\infty} \lambda_k < \infty, \qquad \text{where} \qquad \lambda_k = \frac{M_{k-1}}{M_k}, \quad k = 0, 1, \ldots \tag{1}$$

Lemma 0.1. *For any sequence of positive numbers $\{M_i\}_{i=1}^{\infty}$ satisfying (1), there exists an even function $u\colon \mathbb{R} \to \langle 0; +\infty)$ such that $\operatorname{supp} u = \langle -1; 1\rangle$, $u(x) > 0$ for $x \in (-1; 1)$, $u \in C^\infty(\mathbb{R})$, $\|u^{(n)}\|_\infty \leqslant \frac{1}{2}C(C+1)^2(2C)^n M_n$ for $n \in \mathbb{N} \cup \{0\}$, and moreover, $\hat{u}(x) \geqslant 0$, $x \in \mathbb{R}$, $\int_{-\infty}^{\infty} \hat{u}(x)\,\mathrm{d}x \geqslant \frac{\pi}{C}$.*

This lemma can be used for $M_n = (n!)^{1+\varepsilon}, \varepsilon > 0$.

P r o o f. We shall first construct an even function $g\colon \mathbb{R} \to \langle 0; +\infty)$, nonincreasing on $\langle 0; +\infty)$, $\operatorname{supp} g = \langle -C; C\rangle$, $g \in C^\infty(\mathbb{R})$, $\|g^{(n)}\|_\infty \leqslant \frac{1}{2}(C+1)M_n$, $n \in \mathbb{N} \cup \{0\}$ (cf. [3], chap. 19, ex. 10).

Let the sequence $\{g_n\}_{n=0}^{\infty}$, $g_n\colon \mathbb{R} \to \langle 0; +\infty)$, be defined inductively by the relation

$$g_n(x) = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}(x-t)\,\mathrm{d}t, \quad n \in \mathbb{N}. \tag{2}$$

If $g_{n-1} \in C^{(k)}(\mathbb{R})$, $k \in \mathbb{N} \cup \{0\}$, then $g_n \in C^{(k+1)}(\mathbb{R})$ and

$$g_n^{(l)}(x) = \frac{1}{2\lambda_n}(g_{n-1}^{(l-1)}(x+\lambda_n) - g_{n-1}^{(l-1)}(x-\lambda_n)), \quad l = 1, \ldots, k+1. \tag{3}$$

The relations (2) and (3) imply

$$g_n^{(l)}(x) = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}^{(l)}(x-t)\,\mathrm{d}t, \quad l = 0, 1, \ldots, k. \tag{4}$$

Put $g_0(x) = \max(1 - |x|, 0)$, $x \in \mathbb{R}$. Since $g_0 \in C^{(0)}(\mathbb{R}) = C(\mathbb{R})$, we have $g_n \in C^{(n)}(\mathbb{R})$, $n \in \mathbb{N}$, and the relations (3) and (4) are satisfied with $k = n - 1$.

If g_{n-1} is even, then g_n is even, as $g_n(-x) = \frac{1}{2\lambda_n}\int_{-\lambda_n}^{\lambda_n} g_{n-1}(-x-t)\,\mathrm{d}t = \frac{1}{2\lambda_n}\int_{-\lambda_n}^{\lambda_n} g_{n-1}(x+t)\,\mathrm{d}t = \frac{1}{2\lambda_n}\int_{-\lambda_n}^{\lambda_n} g_{n-1}(x-t)\,\mathrm{d}t = g_n(x)$, $x \in \mathbb{R}$. Since g_0 is even, we have by induction g_n is even, $n \in \mathbb{N}$.

Let $\operatorname{supp} g_{n-1} = \langle -s_{n-1}; s_{n-1}\rangle$, $g_{n-1}(x) > 0$ for $x \in (-s_{n-1}, s_{n-1})$, $s_{n-1} > 0$. Then $g_n(x) = \frac{1}{2\lambda_n}\int_{-\lambda_n}^{\lambda_n} g_{n-1}(x-t)\,\mathrm{d}t = 0$ for $|x| > s_{n-1}+\lambda_n$, $g_n(x) > 0$ for $|x| < s_{n-1}+\lambda_n$, and so $\operatorname{supp} g_n = \langle -s_{n-1}-\lambda_n; s_{n-1}+\lambda_n\rangle$. Since $\operatorname{supp} g_0 = \langle -1; 1\rangle$, $g_0(x) > 0$ for $x \in (-1; 1)$, we have $\operatorname{supp} g_n = \Big\langle -\sum\limits_{k=0}^{n}\lambda_k; \sum\limits_{k=0}^{n}\lambda_k\Big\rangle$, $g_n(x) > 0$ for $x \in \Big(-\sum\limits_{k=0}^{n}\lambda_k; \sum\limits_{k=0}^{n}\lambda_k\Big)$, $n \in \mathbb{N}$.

Let g_{n-1} be nonincreasing on $\langle 0; +\infty)$. Then g_n is nonincreasing on $\langle 0; +\infty)$, because $g_n'(x) = \frac{1}{2\lambda_n}(g_{n-1}(x+\lambda_n) - g_{n-1}(x-\lambda_n)) \leqslant 0$ for $x \geqslant \lambda_n$, $g_n'(x) = \frac{1}{2\lambda_n}(g_{n-1}(x+\lambda_n) - g_{n-1}(\lambda_n - x)) \leqslant 0$ for $x \in \langle 0; \lambda_n\rangle$ by (3). As g_0 is nonincreasing on $\langle 0; +\infty)$, we obtain that g_n is nonincreasing on $\langle 0; +\infty)$ for $n \in \mathbb{N}$.

Now, we prove by induction that $g_k^{(n)}$ is Lipschitzian with the constant M_n for all $k \geqslant n$, $n \in \mathbb{N}\cup\{0\}$.

1) $g_n^{(n)}$ is Lipschitzian with the constant M_n: This is clear for $n = 0$. Let

$$|g_{n-1}^{(n-1)}(x) - g_{n-1}^{(n-1)}(x')| \leqslant M_{n-1}|x - x'|, \quad x, x' \in \mathbb{R}.$$

Then by (3)

$$\begin{aligned}
|g_n^{(n)}(x) - g_n^{(n)}(x')| &\leqslant \frac{1}{2\lambda_n}|g_{n-1}^{(n-1)}(x+\lambda_n) - g_{n-1}^{(n-1)}(x'+\lambda_n)| \\
&\quad + \frac{1}{2\lambda_n}|g_{n-1}^{(n-1)}(x-\lambda_n) - g_{n-1}^{(n-1)}(x'-\lambda_n)|) \\
&\leqslant \frac{2M_{n-1}|x-x'|}{2\lambda_n} = M_n|x - x'|.
\end{aligned}$$

2) Let $k - 1 \geqslant n$ and $|g_{k-1}^{(n)}(x) - g_{k-1}^{(n)}(x')| \leqslant M_n|x - x'|$, $x, x' \in \mathbb{R}$.

Then by (4), $|g_k^{(n)}(x) - g_k^{(n)}(x')| = \frac{1}{2\lambda_k}\int_{-\lambda_k}^{\lambda_k}(g_{k-1}^{(n)}(x-t) - g_{k-1}^{(n)}(x'-t))\,\mathrm{d}t| \leqslant \frac{1}{2\lambda_k}\int_{-\lambda_k}^{\lambda_k} M_n|x-x'|\,\mathrm{d}t = M_n|x - x'|$.

Let $k, l \in \mathbb{N}\cup\{0\}$, $k > l$. Then by (4),

$$\begin{aligned}
|g_k^{(l)}(x) - g_{k-1}^{(l)}(x)| &\leqslant \frac{1}{2\lambda_k}\int_{-\lambda_k}^{\lambda_k}|g_{k-1}^{(l)}(x-t) - g_{k-1}^{(l)}(x)|\,\mathrm{d}t \\
&\leqslant \frac{M_l}{2\lambda_k}\int_{-\lambda_k}^{\lambda_k}|t|\,\mathrm{d}t = \frac{M_l}{2}\lambda_k.
\end{aligned} \tag{5}$$

Let $n \in \mathbb{N} \cup \{0\}$. Since $\frac{M_l}{2} \sum\limits_{k=n+1}^{\infty} \lambda_k < \infty$ for $l = 0, 1, \ldots, n$, the series

$$\tilde{g}_n^{(l)}(x) = \sum_{k=n+1}^{\infty} (g_k^{(l)}(x) - g_{k-1}^{(l)}(x)), \quad l = 0, 1, \ldots, n, \tag{6}$$

converges uniformly in $\mathbb{R}$. It means that for the function $\tilde{g}_n$, defined by (6) for $l = 0$, we have $\tilde{g}_n \in C^{(n)}(\mathbb{R})$. Put

$$\begin{aligned} g(x) = \lim_{n\to\infty} g_n(x) &= \sum_{k=n+1}^{\infty} (g_k(x) - g_{k-1}(x)) + g_n(x) \\ &= \tilde{g}_n(x) + g_n(x), \quad n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{7}$$

Since $g_n \in C^{(n)}(\mathbb{R})$, $\tilde{g}_n \in C^{(n)}(\mathbb{R})$ for any $n \in \mathbb{N} \cup \{0\}$, we have $g \in C^{\infty}(\mathbb{R})$ for $g = g_n + \tilde{g}_n$. By (3) for $l = n$ we obtain

$$\|g_n^{(n)}\|_\infty \leqslant \frac{1}{2\lambda_n}(\|g_{n-1}^{(n-1)}\|_\infty + \|g_{n-1}^{(n-1)}\|_\infty) = \frac{1}{\lambda_n}\|g_{n-1}^{(n-1)}\|_\infty = \frac{\|g_0\|_\infty}{\lambda_n \ldots \lambda_1} = M_n. \tag{8}$$

Further, by (6) and (5) for $l = n$ we get

$$\|\tilde{g}_n^{(n)}\|_\infty \leqslant \frac{M_n}{2} \sum_{k=n+1}^{\infty} \lambda_k. \tag{9}$$

It follows by (7) from (8) and (9) that

$$\|g^{(n)}\|_\infty \leqslant \frac{M_n}{2}\left(2 + \sum_{k=n+1}^{\infty} \lambda_k\right) \leqslant \frac{M_n}{2}(C+1). \tag{10}$$

The function g is even and nonincreasing on $\langle 0; +\infty)$ as the limit of a sequence of functions g_n with the same properties. Obviously $g(x) = 0$ for $|x| \geqslant C = \sum\limits_{k=0}^{\infty} \lambda_k$. We have $g(x_0) > 0$ for $x_0 \in \langle 0; C)$. Indeed, find $n \in \mathbb{N}$ such that $\sum\limits_{k=n+1}^{\infty} \lambda_k < \frac{1}{2}(C - x_0)$. Then $g_n(\frac{C+x_0}{2}) > 0$ and $g_m(x) \geqslant g_n\left(\frac{C+x_0}{2}\right)$ for $m \geqslant n$ and $x \in \left\langle 0; \frac{C+x_0}{2} - \sum\limits_{k=n+1}^{m} \lambda_k \right\rangle$. Hence $g_m(x_0) \geqslant g_n\left(\frac{C+x_0}{2}\right)$ for $m \geqslant n$ and $g(x_0) = \lim\limits_{m\to\infty} g_m(x_0) \geqslant g_n\left(\frac{C+x_0}{2}\right)$. So we have $\operatorname{supp} g = \langle -C; C\rangle$.

Using Fubini's theorem for the nonnegative function $g_{n-1}(x-t)$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} g_n(x)\,\mathrm{d}x &= \frac{1}{2\lambda_n}\int_{-\lambda_n}^{\lambda_n}\left(\int_{-\infty}^{\infty} g_{n-1}(x-t)\,\mathrm{d}x\right)\mathrm{d}t = \int_{-\infty}^{\infty} g_{n-1}(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} g_0(x)\,\mathrm{d}x = 1 \end{aligned}$$

and by the uniform convergence of the series in (6) and by (7) we get

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = \int_{-C}^{C} g(x)\,\mathrm{d}x = 1. \tag{11}$$

The function $g*g(x) = \int_{-\infty}^{\infty} g(t)g(x-t)\,\mathrm{d}t$ is even, nonincreasing on $\langle 0;+\infty)$, positive on $(-2C;2C)$. Further, $\operatorname{supp} g*g = \langle -2C;2C\rangle$, $g*g \in C^{\infty}(\mathbb{R})$ and by (10)

$$\begin{aligned}\|(g*g)^{(n)}\|_{\infty} &= \sup_{x\in\mathbb{R}} |(g*g)^{(n)}(x)| = \sup_{x\in\mathbb{R}}\left|\int_{-\infty}^{\infty} g(t)g^{(n)}(x-t)\,\mathrm{d}t\right| \\ &\leqslant 2C\|g\|_{\infty}\|g^{(n)}\|_{\infty} \leqslant \frac{C}{2}(C+1)^2 M_n.\end{aligned}$$

Put $u(x) = g*g(2Cx)$. Then u is even, nonincreasing on $\langle 0;+\infty)$, $u \in C^{\infty}(\mathbb{R})$, $\operatorname{supp} u = \langle -1;1\rangle$, $u(x) > 0$ for $x \in (-1;1)$, $\|u^{(n)}\|_{\infty} \leqslant \frac{1}{2}C(C+1)^2(2C)^n M_n$, $\hat{u}(x) = \frac{1}{2C}\hat{g}^2(\frac{x}{2C}) \geqslant 0$, $x \in \mathbb{R}$. Since $g \in L^2(\mathbb{R})$ we can use Plancherel's theorem to prove $\int_{-\infty}^{\infty} \hat{u}(x)\,\mathrm{d}x = \frac{1}{2C}\int_{-\infty}^{\infty} \hat{g}^2(\frac{x}{2C})\,\mathrm{d}x = \int_{-\infty}^{\infty} \hat{g}^2(t)\,\mathrm{d}t = 2\pi\int_{-\infty}^{\infty} g^2(t)\,\mathrm{d}t = 2\pi\int_{-C}^{C} g^2(t)\,\mathrm{d}t \geqslant \frac{2\pi}{2C}\left(\int_{-C}^{C} g(t)\,\mathrm{d}t\right)^2 = \frac{\pi}{C}$ by (11) and the Hölder inequality. □

Theorem 0.1. *Let* $m,n \in \mathbb{N}$, $a_k \in \mathbb{R}$, $k = 1,\dots,m$. *Let further* $A(x)$ *be a nonnegative nondecreasing function defined for* $x \in \langle 0;+\infty)$ *and let the integral* $f(s) = \int_0^{\infty} A(x)\mathrm{e}^{-xs}\,\mathrm{d}x$, $s = \sigma + \mathrm{i}t$, *converge for* $\sigma > 1$. *Let the function* $g(s) = f(s) - \sum\limits_{k=1}^{m} \frac{a_k}{(s-1)^k}$ *be continuous in the halfplane* $\sigma \geqslant 1$. *Define* $g_0(t) = g(1+\mathrm{i}t)$, $t \in \mathbb{R}$. *If the function* g_0 *satisfies the conditions:* $g_0^{(i)}$ *is absolutely continuous on* $\mathbb{R}$, $\lim\limits_{\lambda\to+\infty} \frac{1}{\lambda^{n-i}}\int_0^{\lambda} |g_0^{(i)}(t)|\,\mathrm{d}t = 0$ *for* $i = 0,1,\dots,n-1$ *and* $g_0^{(n)} \in L^1(\mathbb{R})$, *then*

$$\left|\mathrm{e}^{-x}A(x) - \sum_{k=0}^{m-1}\frac{a_{k+1}}{k!}x^k\right| \leqslant \frac{2\|g_0^{(n)}\|_1}{\pi x^n}, \quad x > 0.$$

P r o o f. We will prove the theorem in two steps. First, we shall estimate the expression $\mathrm{e}^{x}A(x) - \sum\limits_{k=0}^{m-1}\frac{a_{k+1}}{k!}x^k$ from above (cf. (18)), and, secondly, from below (cf. (23)). We have $f(s) = \int_0^{\infty} A(x)\mathrm{e}^{-xs}\,\mathrm{d}x$, $\frac{(k-1)!}{(s-1)^k} = \int_0^{\infty} x^{k-1}\mathrm{e}^{-(s-1)x}\,\mathrm{d}x$, $k \in \mathbb{N}$, for $\sigma > 1$. Hence $g(s) = f(s) - \sum\limits_{k=1}^{m}\frac{a_k}{(s-1)^k} = \int_0^{\infty}\Big(B(x) - \sum\limits_{k=1}^{m}\frac{a_k x^{k-1}}{(k-1)!}\Big)\mathrm{e}^{-(s-1)x}\,\mathrm{d}x = \int_0^{\infty}\Big(B(x) - \sum\limits_{k=0}^{m-1}\frac{a_{k+1}}{k!}x^k\Big)\mathrm{e}^{-(s-1)x}\,\mathrm{d}x$, where $B(x) = \mathrm{e}^{-x}A(x)$. Take any sequence $\{M_i\}_{i=1}^{\infty}$ of positive numbers, $M_{-1} = M_0 = 1$, such that $C = \sum\limits_{i=0}^{\infty}\frac{M_{i-1}}{M_i} < \infty$. By the preceding lemma there exists a kernel $u \in C^{\infty}(\mathbb{R})$ with $\operatorname{supp} u = \langle -1;1\rangle$ such that u and $\hat{u}$ are even, $u(x) > 0$, $x \in (-1;1)$, $\hat{u}(x) \geqslant 0$, $x \in \mathbb{R}$, $\int_{-\infty}^{\infty} \hat{u}(x)\,\mathrm{d}x \geqslant \frac{\pi}{C}$, $\|u^{(n)}\|_\infty \leqslant D\beta^n M_n$, $n \in \mathbb{N} \cup \{0\}$, where $D = \frac{1}{2}C(C+1)^2$, $\beta = 2C$. Put $g_\varepsilon(t) = g(1+\varepsilon+\mathrm{i}t)$, $t \in \mathbb{R}$, $\varepsilon > 0$. For $\lambda > 0$, $y > 0$ we have

$$\int_{-\lambda}^{\lambda} g_\varepsilon(t) u\Big(\frac{t}{\lambda}\Big)\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t = \int_{-\lambda}^{\lambda} u\Big(\frac{t}{\lambda}\Big)\mathrm{e}^{\mathrm{i}yt}\Big(\int_0^\infty \Big(B(x) - \sum_{k=0}^{m-1} a_{k+1}\frac{x^k}{k!}\Big)\mathrm{e}^{-(\varepsilon+\mathrm{i}t)x}\,\mathrm{d}x\Big)\,\mathrm{d}t.$$

We want to change the order of integration in the last integral: this is possible as for $[t,x] \in \Omega = (-\lambda;\lambda) \times (0;+\infty)$ we have $|u(\frac{t}{\lambda})\mathrm{e}^{\mathrm{i}yt}x^k\mathrm{e}^{-(\varepsilon+it)x}| \leqslant Dx^k\mathrm{e}^{-\varepsilon x} \in L^1(\Omega)$, $k = 0,1,\dots,m-1$. For $s > 1$, $x > 0$, the relation $f(s) = \int_0^\infty A(u)\mathrm{e}^{-us}\,\mathrm{d}u \geqslant A(x)\int_x^\infty \mathrm{e}^{-us}\,\mathrm{d}u = s^{-1}A(x)\mathrm{e}^{-xs}$ holds. It means $A(x) \leqslant sf(s)\mathrm{e}^{sx}$, $B(x) \leqslant sf(s)\mathrm{e}^{(s-1)x}$, $s > 1$, $x > 0$. In particular, $B(x) \leqslant (1+\frac{\varepsilon}{2})f(1+\frac{\varepsilon}{2})\mathrm{e}^{\frac{\varepsilon}{2}x}$, $x > 0$. It follows that $|u(\frac{t}{\lambda})\mathrm{e}^{\mathrm{i}yt}B(x)\mathrm{e}^{-(\varepsilon+\mathrm{i}t)x}| \leqslant (1+\frac{\varepsilon}{2})f(1+\frac{\varepsilon}{2})\mathrm{e}^{-\frac{\varepsilon}{2}x}\cdot D \in L^1(\Omega)$. Consequently, $u(\frac{t}{\lambda})\mathrm{e}^{\mathrm{i}yt}\Big(B(x) - \sum\limits_{k=0}^{m-1} a_{k+1}\frac{x^k}{k!}\Big)\mathrm{e}^{-(\varepsilon+\mathrm{i}t)x}$ is in $L^1(\Omega)$. Hence we can change the order of integration to obtain

$$\begin{aligned}\int_{-\lambda}^{\lambda} g_\varepsilon(t) u\Big(\frac{t}{\lambda}\Big)\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t &= \int_0^\infty \Big(B(x) - \sum_{k=0}^{m-1} a_{k+1}\frac{x^k}{k!}\Big)\mathrm{e}^{-\varepsilon x}\Big(\int_{-\lambda}^{\lambda} \mathrm{e}^{\mathrm{i}(y-x)t}u\Big(\frac{t}{\lambda}\Big)\,\mathrm{d}t\Big)\,\mathrm{d}x\\ &= \int_0^\infty \Big(B(x) - \sum_{k=0}^{m-1} a_{k+1}\frac{x^k}{k!}\Big)\mathrm{e}^{-\varepsilon x}\cdot\lambda\hat{u}(\lambda(y-x))\,\mathrm{d}x.\end{aligned}$$

We need to let $\varepsilon \to 0+$ in this relation. As g_ε tends uniformly to g_0 in $(-\lambda;\lambda)$, we have $\lim\limits_{\varepsilon\to 0+}\int_{-\lambda}^{\lambda} g_\varepsilon(t)u(\frac{t}{\lambda})\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t = \int_{-\lambda}^{\lambda} g_0(t)u(\frac{t}{\lambda})\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t \in \mathbb{R}$. Levi's theorem yields

$$\lim_{\varepsilon\to 0+}\int_0^\infty B(x)\lambda\hat{u}\big(\lambda(y-x)\big)\mathrm{e}^{-\varepsilon x}\,\mathrm{d}x = \lambda\int_0^\infty B(x)\hat{u}\big(\lambda(y-x)\big)\,\mathrm{d}x \in \mathbb{R}\cup\{+\infty\},$$

and Lebesgue's theorem implies

$$\lim_{\varepsilon\to 0+}\int_0^\infty x^k\mathrm{e}^{-\varepsilon x}\lambda\hat{u}(\lambda(y-x))\,\mathrm{d}x = \lambda\int_0^\infty x^k\hat{u}(\lambda(y-x))\,\mathrm{d}x \in \mathbb{R},$$

because

$$\hat{u}(t) \leqslant 2D\beta^n M_n|t|^{-n}, \quad t \neq 0,\ n \in \mathbb{N}. \tag{12}$$

This fact implies that $\int_0^\infty B(x)\hat{u}(\lambda(y-x))\,\mathrm{d}x \in \mathbb{R}$, too, and

$$\begin{aligned}\int_{-\lambda}^{\lambda} g_0(t)u\Big(\frac{t}{\lambda}\Big)\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t &= \lambda\int_0^\infty\Big(B(x) - \sum_{k=0}^{m-1} a_{k+1}\frac{x^k}{k!}\Big)\hat{u}(\lambda(y-x))\,\mathrm{d}x\\ &= \int_{-\infty}^{\lambda y}\Big(B\Big(y-\frac{v}{\lambda}\Big) - \sum_{k=0}^{m-1}\frac{a_{k+1}}{k!}\Big(y-\frac{v}{\lambda}\Big)^k\Big)\hat{u}(v)\,\mathrm{d}v.\end{aligned} \tag{13}$$

Integration by parts yields $\int_{-\lambda}^{\lambda} g_0(t)u(\frac{t}{\lambda})\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t = (\frac{\mathrm{i}}{y})^n \int_{-\lambda}^{\lambda}[g_0(t)u(\frac{t}{\lambda})]^{(n)}\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t = (\frac{\mathrm{i}}{y})^n \int_{-\lambda}^{\lambda}\Big[\sum\limits_{k=0}^{n}\binom{n}{k}u^{(n-k)}(\frac{t}{\lambda})\cdot\frac{g_0^{(k)}(t)}{\lambda^{n-k}}\Big]\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t$. Then, by the properties of the kernel u, we obtain

$$\left|\int_{-\lambda}^{\lambda} g_0(t)u(\frac{t}{\lambda})\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t\right| \leqslant 2Dy^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{\beta^{n-k}M_{n-k}}{\lambda^{n-k}}\int_0^{\lambda}|g_0^{(k)}(t)|\,\mathrm{d}t,\ y>0. \tag{14}$$

Let $0 < a \leqslant \lambda y$. For $v \in \langle -a; a\rangle \subset (-\infty; \lambda y\rangle$ we have by monotonicity of the function A

$$\begin{aligned}&A\Big(y-\frac{a}{\lambda}\Big) \leqslant A\Big(y-\frac{v}{\lambda}\Big) \leqslant A\Big(y+\frac{a}{\lambda}\Big), \mathrm{e}^{y-\frac{a}{\lambda}}B\Big(y-\frac{a}{\lambda}\Big) \leqslant \mathrm{e}^{y-\frac{v}{\lambda}}B\Big(y-\frac{v}{\lambda}\Big)\\ &\leqslant \mathrm{e}^{y+\frac{a}{\lambda}}B\Big(y+\frac{a}{\lambda}\Big), \text{ i.e. } \mathrm{e}^{-\frac{2a}{\lambda}}B\Big(y-\frac{a}{\lambda}\Big) \leqslant B\Big(y-\frac{v}{\lambda}\Big), \quad v \in \langle -a; a\rangle.\end{aligned}$$

From this inequality we deduce

$$\mathrm{e}^{-\frac{2a}{\lambda}}B\Big(y-\frac{a}{\lambda}\Big)\int_{-a}^{a}\hat{u}(v)\,\mathrm{d}v \leqslant \int_{-a}^{a}B\Big(y-\frac{v}{\lambda}\Big)\hat{u}(v)\,\mathrm{d}v \leqslant \int_{-\infty}^{\lambda y}B\Big(y-\frac{v}{\lambda}\Big)\hat{u}(v)\,\mathrm{d}v.$$

It follows easily from these inequalities that

$$\begin{aligned}\mathrm{e}^{-\frac{2a}{\lambda}}B\Big(y-\frac{a}{\lambda}\Big)\int_{-a}^{a}\hat{u}(v)\,\mathrm{d}v - \int_{-\infty}^{\lambda y}\sum_{k=0}^{m-1}\frac{a_{k+1}}{k!}(y-\frac{v}{\lambda})^k\hat{u}(v)\,\mathrm{d}v&\\ \leqslant \int_{-\infty}^{\lambda y}\Big(B\Big(y-\frac{v}{\lambda}\Big) - \sum_{k=0}^{m-1}\frac{a_{k+1}}{k!}\Big(y-\frac{v}{\lambda}\Big)^k\Big)\hat{u}(v)\,\mathrm{d}v.&\end{aligned}$$

Using the relation (13) and the estimate (14) we finally obtain

$$\begin{aligned}\mathrm{e}^{-\frac{2a}{\lambda}}B\Big(y-\frac{a}{\lambda}\Big)\int_{-a}^{a}\hat{u}(v)\,\mathrm{d}v - \int_{-\infty}^{\lambda y}\sum_{k=0}^{m-1}\frac{a_{k+1}}{k!}\Big(y-\frac{v}{\lambda}\Big)^k\hat{u}(v)\,\mathrm{d}v&\\ \leqslant 2Dy^{-n}\sum_{k=0}^{n}\binom{n}{k}\beta^{n-k}M_{n-k}\cdot\frac{1}{\lambda^{n-k}}\int_0^{\lambda}|g_0^{(k)}(t)|\,\mathrm{d}t.&\end{aligned} \tag{15}$$

We shall show that for every $k \in \mathbb{N}\cup\{0\}$

$$\lim_{\lambda\to+\infty}\int_{-\infty}^{\lambda y}\Big(y-\frac{v}{\lambda}\Big)^k\hat{u}(v)\,\mathrm{d}v = y^k\int_{-\infty}^{\infty}\hat{u}(v)\,\mathrm{d}v. \tag{16}$$

Note that the relation (12) implies $|v|^n\hat{u}(v) \in L^1(\mathbb{R})$ for all $n \in \mathbb{N}\cup\{0\}$. Then $\lim\limits_{\lambda\to+\infty}\int_{-\infty}^{\lambda y}(y-\frac{v}{\lambda})^k\hat{u}(v)\,\mathrm{d}v = \lim\limits_{\lambda\to+\infty}\int_{-\infty}^{0}(y-\frac{v}{\lambda})^k\hat{u}(v)\,\mathrm{d}v + \lim\limits_{\lambda\to\infty}\int_0^{\infty}f_\lambda(v)\,\mathrm{d}v$, where

$f_\lambda(v) = [\max(y - \frac{v}{\lambda}, 0)]^k \hat{u}(v)$. Since $0 \leqslant f_\lambda(v) \leqslant y^k \hat{u}(v) \in L^1((0,+\infty))$, we have $\lim\limits_{\lambda\to+\infty} \int\limits_0^\infty f_\lambda(v)\,\mathrm{d}v = y^k \int_0^\infty \hat{u}(v)\,\mathrm{d}v$ by Lebesgue's theorem. Levi's theorem gives $\lim\limits_{\lambda\to+\infty} \int_{-\infty}^0 (y - \frac{v}{\lambda})^k \hat{u}(v)\,\mathrm{d}v = y^k \int_{-\infty}^0 \hat{u}(v)\,\mathrm{d}v$. The relation (16) is proved.

Let $\lambda \to +\infty$, $a \to +\infty$, in such a way that $\frac{a}{\lambda} \to 0+$ in the relation (15). From the assumptions of the theorem we obtain by (16)

$$\lim_{x\to y-} B(x) \cdot \int_{-\infty}^\infty \hat{u}(v)\,\mathrm{d}v - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} y^k \int_{-\infty}^\infty \hat{u}(v)\,\mathrm{d}v \leqslant D y^{-n} \|g_0^{(n)}\|_1. \tag{17}$$

Since the function A is nondecreasing on $\langle 0; +\infty)$, the proper limits $\lim\limits_{x\to y-} A(x)$, $\lim\limits_{x\to y+} A\ (x)$ exist for $y > 0$ and $\lim\limits_{x\to y-} A(x) \leqslant A(y) \leqslant \lim\limits_{x\to y+} A(x)$, $y > 0$. It follows that $\lim\limits_{x\to y-} B(x) \leqslant B(y) \leqslant \lim\limits_{x\to y+} B(x)$, $y > 0$. The functions A and B are continuous on $\langle 0; +\infty)$ except for a countable set of points. The inequality

$$\Bigl[B(y) - \sum_{k=0}^{m-1} \frac{a_{k+1}}{k!} y^k\Bigr] \int_{-\infty}^\infty \hat{u}(v)\,\mathrm{d}v \leqslant D y^{-n} \|g_0^{(n)}\|_1 \tag{18}$$

holds, however, at the points of discontinuity of the function B as well. We can verify this fact letting $y \to y_1+$, where y_1 is a point of discontinuity and y's are points of continuity of the function B.

We know by (18) that there exists a polynomial $\tilde{P}$ with nonnegative coefficients, $\deg \tilde{P} \leqslant m - 1$, such that $0 \leqslant B(y) \leqslant \tilde{P}(y)$ for $y > 0$. So we have for $\lambda > 0$, $y > 0$, $1 \leqslant a \leqslant \lambda y$

$$\begin{aligned}\int_{-\infty}^{\lambda y} B\Bigl(y - \frac{v}{\lambda}\Bigr)\hat{u}(v)\,\mathrm{d}v \leqslant \int_{-\infty}^{-a} \tilde{P}\Bigl(y - \frac{v}{\lambda}\Bigr)\hat{u}(v)\,\mathrm{d}v + \int_a^{\lambda y} \tilde{P}\Bigl(y - \frac{v}{\lambda}\Bigr)\hat{u}(v)\,\mathrm{d}v \\ + \int_{-a}^a B\Bigl(y - \frac{v}{\lambda}\Bigr)\hat{u}(v)\,\mathrm{d}v.\end{aligned} \tag{19}$$

There exists $K = K(y) > 0$ such that

$$0 \leqslant \tilde{P}\Bigl(y - \frac{v}{\lambda}\Bigr)\hat{u}(v) \leqslant \frac{K}{v^2} \tag{20}$$

for $v \in (-\infty; -a\rangle \cup \langle a; \lambda y\rangle$. For $v \in \langle -a; a\rangle$ we have

$$B\Bigl(y - \frac{v}{\lambda}\Bigr) \leqslant \mathrm{e}^{\frac{2a}{\lambda}} B\Bigl(y + \frac{a}{\lambda}\Bigr). \tag{21}$$

The relations (19), (20) and (21) yield

$$\mathrm{e}^{\frac{2a}{\lambda}} B\Big(y+\frac{a}{\lambda}\Big)\int_{-a}^{a}\hat{u}(v)\,\mathrm{d}v \geqslant \int_{-a}^{a} B\Big(y-\frac{v}{\lambda}\Big)\hat{u}(v)\,\mathrm{d}v$$
$$\geqslant \int_{-\infty}^{\lambda y} B\Big(y-\frac{v}{\lambda}\Big)\hat{u}(v)\,\mathrm{d}v - 2\int_{a}^{\infty}\frac{K}{v^2}\,\mathrm{d}v.$$

Using (13) a (14) we can deduce from this fact that

$$\begin{aligned}\mathrm{e}^{\frac{2a}{\lambda}} B\Big(y+\frac{a}{\lambda}\Big)\int_{-a}^{a}\hat{u}(v)\,\mathrm{d}v - \int_{-\infty}^{\lambda y}\sum_{k=0}^{m-1}\frac{a_{k+1}}{k!}\Big(y-\frac{v}{\lambda}\Big)^k\hat{u}(v)\,\mathrm{d}v + \frac{2K}{a}\\ \geqslant -2Dy^{-n}\sum_{k=0}^{n}\binom{n}{k}\beta^{n-k}\frac{M_{n-k}}{\lambda^{n-k}}\int_0^{\lambda}|g_0^{(k)}(t)|\,\mathrm{d}t.\end{aligned}\tag{22}$$

Letting in (22) $\lambda\to+\infty$, $a\to+\infty$, in such a way that $\frac{a}{\lambda}\to 0+$, we obtain

$$\Big[B(y)-\sum_{k=0}^{m-1}\frac{a_{k+1}}{k!}y^k\Big]\int_{-\infty}^{\infty}\hat{u}(v)\,\mathrm{d}v \geqslant -Dy^{-n}\|g_0^{(n)}\|_1,\tag{23}$$

provided $y>0$ is a point of continuity of the function B. Letting in (23) $y\to y_1-$, where y_1 is a point of discontinuity and y's are points of continuity of the function B, we verify validity of (23) at the points of discontinuity of $B(x)$. Finally, $D(\int_{-\infty}^{\infty}\hat{u}(v)\,\mathrm{d}v)^{-1}\leqslant\frac{C}{2}(C+1)^2\cdot\frac{C}{\pi}=\frac{C^2(C+1)^2}{2\pi}$. Since $C=\sum\limits_{n=0}^{\infty}\frac{M_{n-1}}{M_n}>1$ can be chosen arbitrarily, the theorem is proved. □

There exist constants $K>0$ and $\beta>1$ such that

$$\Big|\Big[\frac{\xi'}{\xi}(1+\mathrm{i}t)\Big]^{(j)}\Big|\leqslant K\beta^j(j!)^2\log^{9(j+1)}t\tag{24}$$

for all $t\geqslant 3$ and $j=0,1,\ldots$ (cf. [5]). This relation implies

$$\|h_1^{(j)}(1+\mathrm{i}t)\|_1\leqslant\beta^j(j!)^2[9(j+1)]!\tag{25}$$

for all $j=0,1,\ldots$ and a suitable $\beta>1$. Theorem 0.1 now yields

$$|\mathrm{e}^{-x}\psi_1(\mathrm{e}^x)-1|\leqslant\frac{2}{\pi x^n}\beta^n(n!)^2[9(n+1)]!\leqslant x^{-n}\beta_1^n n^{11n}\tag{26}$$

for all $x>0$ and $n\in\mathbb{N}$ with suitable $\beta_1>1$. Minimizing $x^{-n}\beta_1^n n^{11n}$ over $n\in\mathbb{N}$ for a fixed $x>0$ we obtain

$$\psi_1(x)=x+O(x\exp(-2c\log^{\frac{1}{11}}x)),\quad x\to+\infty$$

and

$$\pi(x) = \int_2^{\infty} \frac{\mathrm{d}u}{\log u} + O(x \exp(-c \log^{\frac{1}{11}} x)), \quad x \to +\infty$$

with some $c > 0$. By analytic methods we can obtain an essentially better estimation of the error term in the P.N.T. (cf. [5]).

This theorem generalizes Theorem 2 of [2] in two directions: both the main and the remainder terms are more general. The main term can be found in [4], Chapter 5. The proof of the present theorem is a modification of the proof of Theorem 2 from the book [1], p. 124.

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Author's address: Department of Mathematics, University of West Bohemia, Univerzitní 22, 306 14 Plzeň, Czech Republic.