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AN ALGEBRAIC CHARACTERIZATION OF GEODETIC GRAPHS[1]

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Abstract. We say that a binary operation $*$ is associated with a (finite undirected) graph G (without loops and multiple edges) if $*$ is defined on $V(G)$ and $uv \in E(G)$ if and only if $u \neq v$, $u * v = v$ and $v * u = u$ for any u, $v \in V(G)$. In the paper it is proved that a connected graph G is geodetic if and only if there exists a binary operation associated with G which fulfils a certain set of four axioms. (This characterization is obtained as an immediate consequence of a stronger result proved in the paper).

In the present paper we will prove that a connected graph G is geodetic if and only if there exists a binary operation associated—in a certain sense—with G and fulfilling a set of four axioms. We will obtain this characterization of geodetic graphs as a trivial consequence of a stronger result.

(Note that in [4] and [5] the present author proved that a connected graph G is geodetic if and only if there exists a set of paths in G which fulfils certain axioms.)

In the present paper the letters f, g, h, i, j, k, m and n are reserved for denoting integers.

By a graph we mean a graph in the sense of [1], [2], or [3] (i.e. a finite undirected graph without loops and multiple edges). Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. If $u, v \in V(G)$, then we say that a sequence

$$(w_0, \ldots, w_n), \tag{0}$$

where $n \geqslant 0$, is a $u - v$ path in G if $w_0, \ldots, w_n$ are mutually distinct vertices of G, $u = w_0$, $v = w_n$, and if $n \geqslant 1$, then $w_0w_1, \ldots, w_{n-1}w_n \in E(G)$.

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Let G be a graph, and let $*$ be a binary operation defined on $V(G)$. We will say that $*$ is associated with G if

$$uv \in E(G) \quad \text{if and only if} \quad u \neq v,\ u * v = v \text{ and } v * u = u$$

for any $u, v \in V(G)$. Obviously, every binary operation defined on a finite nonempty set is associated with exactly one graph.

Let G be a connected graph. We denote by d the distance function of G. If $u, v \in V(G)$ and (0) is an $u - v$ path in G, then we say that (0) is a shortest path (or a geodesic [2]) in G if $n = d(u, v)$. We denote by N the mapping of $V(G) \times V(G)$ into the set of all subsets of $V(G)$ defined for every $(u, v) \in V(G) \times V(G)$ as follows:

$$N(u, v) = \{u\} \quad \text{if } u = v, \text{ and}$$
$$N(u, v) = \{w \in V(G);\ d(u, w) = 1 \text{ and } d(w, v) = d(u, v) - 1\} \text{ if } u \neq v.$$

Note that if $u \neq v$, then $N(u, v)$ is the same as $N_1(u, v)$ in the sense of [3].

A graph G is called *geodetic* if it is connected and there exists exactly one shortest $u - v$ path in G for each ordered pair of vertices u and v of G. Proceeding by induction on the distance of vertices, we see that a connected graph G is geodetic if and only if $|N(u, v)| = 1$ for each ordered pair of vertices u and v of G.

Let G be a geodetic graph. We will say that a binary operation $*$ defined on $V(G)$ is the *proper* operation of G if $u * v$ is the only vertex of $N(u, v)$ for each ordered pair of vertices u and v of G. Obviously, every geodetic graph has exactly one proper operation and this operation is associated with it. Moreover, every binary operation is the proper operation of at most one geodetic graph.

We will prove a theorem asserting that if G is a graph and $*$ is a binary operation associated with G, then G is geodetic and $*$ is the proper operation of G if and only if G is connected and the operation $*$ fulfils Axioms I–IV stated below. As a trivial consequence we will obtain the following characterization of geodetic graphs: A connected graph G is geodetic if and only if there exists a binary operation associated with G which fulfils Axioms I–IV.

For proving our theorem we will need five lemmas.

Lemma 1. *Let G be a geodetic graph, and let $*$ be its proper operation. Then the operation $*$ fulfils the following Axioms* I–IV*:*

I *$(u * v) * u = u$ for all $u, v \in V(G)$;*

II *if $(u * v) * v = u$, then $u = v$ for all $u, v \in V(G)$;*

III *if $v * u \neq u$, then $u * (v * u) = u * v$ for all $u, v \in V(G)$;*

IV *if $w * v = v$, then either $u * w = u * v$ or $w * (u * v) = v$ for all $u, v, w \in V(G)$.*

P r o o f. It is very easy to verify Axioms I–III. We will verify Axiom IV. Consider $u, v, w \in V(G)$ such that $w*v = v$ and $u*w \neq u*v$. Then $d(v, w) = 1$. If $u \in \{v, w\}$, then $w*(u*v) = v$. Let $v \neq u \neq w$. Then $d(u, v) = d(u, w)$. Moreover, we see that $d(u*v, w) > d(u*v, v)$. Since $d(v, w) = 1$, we have $d(u*v, w) = d(u*v, v) + 1$. This implies that $w*(u*v) = v$, which completes the proof.

In Lemmas 2–5 we will assume that a graph G and a binary operation $*$ associated with G are given. Moreover, we will assume that the operation $*$ fulfils Axioms I–IV.

□

Lemma 2. *Let r and s be vertices of G. Then*

$$r * s = r \quad \text{if and only if} \quad r = s \tag{1}$$

and

$$r * s = s \quad \text{if and only if} \quad s * r = r. \tag{2}$$

P r o o f. Axioms I and II imply (1). Axiom I implies (2). □

Lemma 3. *Let $h \geqslant 1$, let $r_1, \ldots, r_{h+1}$, s_0, s_1 be vertices of G, let*

$$r_1 \neq r_2, \ldots, r_h \neq r_{h+1}, \tag{3}$$

$$r_1 * s_0 = r_2, \ldots, r_h * s_0 = r_{h+1}, \tag{4}$$

and let

$$s_1 * r_1 = s_0. \tag{5_0}$$

Then

$$r_f * s_1 = r_{f+1} \quad \text{and} \quad s_1 * r_{f+1} = s_0 \tag{5_f}$$

for each f, $1 \leqslant f \leqslant h$.

P r o o f. We will prove that (5_g) holds for each g, $0 \leqslant g \leqslant h$. We proceed by induction on g. We know that (5_0) holds. Let $1 \leqslant g \leqslant h$. According to (5_{g-1}), $s_1 * r_g = s_0$. As follows from (4), $r_g * s_0 = r_{g+1}$.

If $r_g = s_0$, then, by (1), $r_{g+1} = r_g * s_0 = r_g$, which contradicts (3). We have $r_g \neq s_0$ and therefore, $s_1 * r_g \neq r_g$. By virtue of Axiom III,

$$r_g * (s_1 * r_g) = r_g * s_1.$$

Since $s_1 * r_g = s_0$, we have $r_g * s_1 = r_g * s_0 = r_{g+1}$. Since $s_1 * r_g \neq r_g$, (2) implies that $r_g * s_1 \neq s_1$. By Axiom III,

$$s_1 * (r_g * s_1) = s_1 * r_g.$$

Hence $s_1 * r_{g+1} = s_1 * r_g = s_0$. We see that (5_g) holds, which completes the proof of the lemma. $\square$

Lemma 4. *Let $k \geqslant 1$, let $u_0, u_1, \ldots, u_{k+1}$, $v_0, \ldots, v_k$ be vertices of G, let*

$$u_0 \neq u_1, \quad u_1 \neq u_2, \ldots, u_k \neq u_{k+1}, \tag{6}$$

$$u_0 * v_0 = u_1, \quad u_1 * v_0 = u_2, \ldots, u_k * v_0 = u_{k+1}, \tag{7_0}$$

and let

$$v_1 * u_1 = v_0, \ldots, v_k * u_k = v_{k-1}. \tag{8}$$

Then

$$u_i * v_i = u_{i+1}, \ldots, u_k * v_i = u_{k+1} \tag{7_i}$$

for each i, $1 \leqslant i \leqslant k$.

P r o o f. We will prove that (7_j) holds for each j, $0 \leqslant j \leqslant k$. We proceed by induction on j. We know that (7_0) holds. Let $1 \leqslant j \leqslant k$. According to (7_{j-1}),

$$u_j * v_{j-1} = u_{j+1}, \ldots, u_k * v_{j-1} = u_{k+1}.$$

As follows from (8), $v_j * u_j = v_{j-1}$. By (6),

$$u_j \neq u_{j+1}, \ldots, u_k \neq u_{k+1}.$$

Lemma 3 implies that (7_j) holds, which completes the proof of the lemma. $\square$

Lemma 5. *Assume that G is connected. Let $n \geqslant 1$, and let $x_0, x_1, x_2, \ldots$ be vertices of G such that*

$$(x_n, x_{n-1}, \ldots, x_0) \tag{9}$$

is a shortest path in G and

$$x_{n+f+1} = x_{n+f} * x_0 \quad \textit{for each } f \geqslant 0. \tag{10}$$

Then there exists $m \geqslant n$ such that $x_{n+m} = x_0$.

Proof. Combining (10), Axiom I and (2) we get

$$x_{n+f+1} * x_{n+f} = x_{n+f} \quad \text{and} \quad x_{n+f} * x_{n+f+1} = x_{n+f+1} \quad \text{for each } f \geqslant 0. \tag{11}$$

First, we want to prove that there exists $m \geqslant 0$ such that $x_{n+m} = x_0$. Suppose, to the contrary, that

$$x_{n+f} \neq x_0 \quad \text{for each } f \geqslant 0. \tag{12}$$

Combining (10), (12) and (1), we get

$$x_{n+f+1} \neq x_{n+f} \quad \text{for each } f \geqslant 0. \tag{13}$$

Recall that $*$ is a binary operation associated with G and that (9) is a path in G. Combining these facts with (11), we get

$$x_{g+1} * x_g = x_g \quad \text{for each } g \geqslant 0. \tag{14}$$

Recall that $V(G)$ is finite. This implies that there exist h and i such that $n \leqslant h < i$ and $x_i = x_h$. By virtue of (13), $h+2 \leqslant i$. Moreover, using Axiom II we get $h+3 \leqslant i$.

By (10),

$$x_h * x_0 = x_{h+1}, \ldots, x_{i-1} * x_0 = x_i$$

and by (1) and (13),

$$x_h * x_h \neq x_{h+1}.$$

This means that there exists j, $0 \leqslant j \leqslant h-1$, satisfying

$$x_h * x_j = x_{h+1}, \ldots, x_{i-1} * x_j = x_i \tag{15}$$

and there exists k, $h \leqslant k \leqslant i-1$, such that

$$x_k * x_{j+1} \neq x_{k+1}. \tag{16}$$

By (14), $x_{j+1} * x_j = x_j$ and by (15), $x_k * x_j = x_{k+1}$. We have $x_k * x_{j+1} \neq x_k * x_j$. Axiom IV implies that

$$x_{j+1} * (x_k * x_j) = x_j.$$

Hence

$$x_{j+1} * x_{k+1} = x_j. \tag{17}$$

Recall that $x_i = x_h$. It is not difficult to see that (13), (15) and (17) together with Lemma 3 imply that $x_k * x_{j+1} = x_{k+1}$, which contradicts (16).

Thus we have proved that there exists $m \geqslant 0$ such that $x_{n+m} = x_0$. Since (9) is a shortest path in G, we get $d(x_n, x_0) = n$. Since $*$ is associated with G, it follows from (11) that

$$d(x_n, x_{n+f}) \leqslant f \quad \text{for each } f \geqslant 0.$$

Thus $m \geqslant n$, which completes the proof of the lemma.

Now we will state the theorem which is the main result of the present paper. □

Theorem. *Let G be a graph, and let $*$ be a binary operation associated with G. Then the following statements* (A) *and* (B) *are equivalent:*

(A) *G is geodetic and $*$ is its proper operation;*

(B) *G is connected and operation $*$ fulfils Axioms I - IV.*

P r o o f. If (A) holds, then, by virtue of Lemma 1, (B) also holds.

Conversely, let (B) hold. Then G is connected. We will prove that

$$(18_n) \quad N(u,v) = \{u * v\} \quad \text{for arbitrary vertices } u \text{ and } v \text{ of } G \text{ fulfilling } d(u,v) \leqslant n$$

for each $n \geqslant 0$. As follows from (1) and from the fact that $*$ is associated with G, both (18_0) and (18_1) hold.

Let $n \geqslant 2$. If there exist no vertices y and z of G such that $d(y,z) = n$, then (18_n) is a trivial consequence of (18_{n-1}). Assume that there exist such vertices y and z.

Consider arbitrary vertices u and v of G such that $d(u,v) = n$. Obviously, $N(u,v) \neq \emptyset$. Moreover, consider an arbitrary $w \in N(u,v)$. There exist vertices $x_0, \ldots, x_{n-1}$, x_n of G such that $x_0 = v$, $x_{n-1} = w$, $x_n = u$ and (9) is a shortest path in G. Thus $x_{n-1} \in N(x_n, x_0)$. We define vertices x_{n+1}, x_{n+2}, $x_{n+3}, \ldots$ as in (10). By Lemma 5, there exists $m \geqslant n$ such that $x_{n+m} = x_0$. Without loss of generality we assume that

$$x_{n+1} \neq x_0, \ldots, x_{n+m-1} \neq x_0. \tag{19}$$

Recall that $x_{n+1} = u * v$ and $x_{n-1} \in N(u, v)$. We want to prove that $x_{n-1} = x_{n+1}$. Suppose, to the contrary, that

$$x_{n-1} \neq x_{n+1}. \tag{20}$$

We now will show that

$$x_m * x_0 \neq x_{m-1}. \tag{21}$$

If $m = n$, then (21) is an immediate consequence of (20). Let $m > n$. By virtue of (10), $x_m = x_{m-1} * x_0$. If $x_m * x_0 = x_{m-1}$, then

$$(x_m * x_0) * x_0 = x_m$$

and, by Axiom II, $x_m = x_0$, which contradicts (19). Thus (21) holds. We have

$$x_m * x_{n+m} \neq x_{m-1}.$$

This means that there exists k, $0 \leqslant k < m$, such that

$$x_{k+1} * x_{n+k+1} \neq x_k \tag{22}$$

and if $k \geqslant 1$, then

$$x_1 * x_{n+1} = x_0, \ldots, x_k * x_{n+k} = x_{k-1}. \tag{23}$$

Recall that $*$ is associated with G and

$$x_n * x_0 = x_{n+1}, \ldots, x_{n+k} * x_0 = x_{n+k+1}.$$

As follows from (1), (2), (10), (19) and Axiom I,

$$x_n x_{n+1}, \ldots, x_{n+k} x_{n+k+1} \in E(G). \tag{24}$$

Thus $x_n \neq x_{n+1}, \ldots, x_{n+k} \neq x_{n+k+1}$. Lemma 4 implies that if $k \geqslant 1$, then

$$x_{n+1} * x_1 = x_{n+2}, \ldots, x_{n+k} * x_k = x_{n+k+1}.$$

Since $x_n * x_0 = x_{n+1}$, we have

$$x_{n+f} * x_f = x_{n+f+1} \quad \text{for each } f,\ 0 \leqslant f \leqslant k. \tag{25}$$

Recall that $d(x_n, x_0) = n$ and $x_{n-1} \neq x_{n+1}$. There exists j, $0 \leqslant j \leqslant k$, such that

$$d(x_{n+j}, x_j) = n \quad \text{and} \quad x_{n+j-1} \neq x_{n+j+1}, \tag{26}$$

and

$$\text{if } j < k, \quad \text{then either} \quad d(x_{n+j+1}, x_{j+1}) \leqslant n-1 \text{ or } x_{n+j} = x_{n+j+2}. \tag{27}$$

Recall that (9) is a path. Combining this fact with (24), we have $x_j x_{j+1} \in E(G)$.

We distinguish two cases.

Case 1. Let $j < k$. Then $k \geqslant 1$ and by (23),

$$x_{j+1} * x_{n+j+1} = x_j. \tag{28}$$

Assume that $d(x_{n+j+1}, x_{j+1}) \leqslant n-1$. By virtue of (18_{n-1}),

$$N(x_{j+1}, \quad x_{n+j+1}) = \{x_{j+1} * x_{n+j+1}\}.$$

According to (28), $N(x_{j+1}, x_{n+j+1}) = \{x_j\}$. This implies that $d(x_{n+j+1}, x_j) \leqslant n-2$. We have $d(x_{n+j}, x_j) \leqslant 1 + d(x_{n+j+1}, x_j) \leqslant n-1$, which contradicts (26). Hence $d(x_{n+j+1}, x_{j+1}) = n$.

Since $j < k$, it follows from (27) that $x_{n+j} = x_{n+j+2}$. By (10),

$$x_{n+j+2} = (x_{n+j} * x_0) * x_0.$$

Thus $(x_{n+j} * x_0) * x_0 = x_{n+j}$. By virtue of Axiom II, $x_{n+j} = x_0$, which is a contradiction.

Case 2. Let $j = k$. By (22), $x_{j+1} * x_{n+j+1} \neq x_j$. Thus, as follows from (25),

$$x_{j+1} * (x_{n+j} * x_j) \neq x_j. \tag{29}$$

Recall that $x_j x_{j+1} \in E(G)$. This implies that $x_{j+1} * x_j = x_j$. Combining this fact with (29) and Axiom IV, we get

$$x_{n+j} * x_{j+1} = x_{n+j} * x_j.$$

Therefore, by (25),

$$x_{n+j} * x_{j+1} = x_{n+j+1}. \tag{30}$$

According to (26), $d(x_{n+j}, x_j) = n$. Thus $d(x_{n+j}, x_{j+1}) = n - 1$. By virtue of (18_{n-1}),

$$N(x_{n+j}, x_{j+1}) = \{x_{n+j} * x_{j+1}\}. \tag{31}$$

Since $d(x_{n+j}, x_{j+1}) = n - 1 \geqslant 1$, we have $d(x_{n+j-1}, x_{j+1}) = n - 2$. This implies that

$$x_{n+j-1} \in N(x_{n+j}, x_{j+1}).$$

As follows from (31), $x_{n+j-1} = x_{n+j} * x_{j+1}$. By virtue of (30), $x_{n+j-1} = x_{n+j+1}$, which contradicts (26).

We have proved that $w = x_{n-1} = x_{n+1} = u * v$. Thus (18_n) holds. This implies that G is geodetic and $*$ is its proper operation. Hence (A) holds, which completes the proof of the theorem. $\square$

The following corollary is an immediate consequence of our theorem.

Corollary. *Let G be a connected graph. Then G is geodetic if and only if there exists a binary operation $*$ associated with G which fulfils Axioms* I–IV.

Remark 1. Let G be a connected graph. We will say that an ordered triple (u, v, w) of vertices in G is a step in G if $u \neq w$ and $v \in N(u, w)$. An axiomatic characterization of the set of all steps in G was given by the present author in [6]. (Note that some ideas in the present paper were inspired by the paper [6].)

Remark 2. Let G be a graph with exactly $n \geqslant 2$ components, say $G_1, \ldots,$ and G_n. Assume that for each i, $1 \leqslant i \leqslant n$, G_i is a geodetic graph different from a tree. Consider arbitrary j, $1 \leqslant j \leqslant n$. Let $*_j$ denote the proper operation of G_j. Since G_j is connected and different from a tree, there exists a unicyclic graph F_j which spans G_j. Let $C^{(j)}$ denote the cycle of F_j. Orient each edge of F_j in such a way that for the resulting digraph, say H_j, we have:

if $r \in V(F_j)$ and $s \in V(C^{(j)})$, then there exists a directed path from r to s in H_j.

Clearly, for every $u \in V(G_j)$ there exists exactly one vertex $f_j(u)$ of G_j such that $(u, f_j(u))$ is an arc in H_j.

Let $*$ denote a binary operation defined on $V(G)$ as follows. Consider arbitrary $x, y \in V(G)$. Then there exist k, $m \in \{1, \ldots, n\}$ such that $x \in V(G_k)$ and $y \in V(G_m)$. If $k = m$, then we put $x * y = x *_k y$. If $k \neq m$, then we put $x * y = f_k(x)$. It is easy to see that operation $*$ is associated with G and that $*$ fulfils Axioms I–IV.

Thus we see that the assumption that G is connected cannot be deleted from our corollary.

On the other hand, if G is disconnected and at least one of the components of G is a tree, then it is not difficult to show that no binary operation associated with G fulfils Axioms I and II. Therefore, our corollary can be complemented as follows:

Let G be a disconnected graph. Then each component of G is a geodetic graph different from a tree if and only if there exists a binary operation $*$ associated with G which fulfils Axioms I–IV.

Remark 3. If G is a geodetic graph of diameter not exceeding two and $*$ is its proper operation, then $*$ fulfils the following Axiom $\widetilde{\mathrm{II}}$:

$\widetilde{\mathrm{II}}$ $(u * v) * v = v$ for all $u, v \in V(G)$.

Obviously, Axiom II is an immediate consequence of Axiom $\widetilde{\mathrm{II}}$. The following characterization can be easily derived from our corollary:

A graph G is a geodetic graph of diameter not exceeding two if and only if there exists a binary operation $*$ associated with G which fulfils Axioms I, $\widetilde{\mathrm{II}}$, III and IV.

Remark 4. The author's interest in the proper operation of a geodetic graph is connected to his research in semiotics.

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