Czechoslovak Mathematical Journal, 48 (123) (1998), 677–685

NONOSCILLATION AND ASYMPTOTIC BEHAVIOUR FOR THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS

Aydin Tiryaki and A. Okay Çelebi, Ankara

(Received November 8, 1995)

Abstract. In this paper we consider the equation

$$y''' + q(t){y'}^{\alpha} + p(t)h(y) = 0,$$

where p, q are real valued continuous functions on $[0,\infty)$ such that $q(t) \geqslant 0$, $p(t) \geqslant 0$ and $h(y)$ is continuous in $(-\infty,\infty)$ such that $h(y)y > 0$ for $y \neq 0$. We obtain sufficient conditions for solutions of the considered equation to be nonoscillatory. Furthermore, the asymptotic behaviour of these nonoscillatory solutions is studied.

MSC 2000: 34C15, 34D05

Keywords: Third order nonlinear differential equations, nonoscillatory solutions, asymptotic properties of solutions

1. Introduction

Finding sufficient conditions for nonoscillation of solutions is a problem of general interest in the theory of ordinary and delay differential equations ([6], [13], [16]). In this paper, we consider

$$y''' + q(t){y'}^{\alpha} + p(t)h(y) = 0, \tag{1}$$

where p, q are real valued continuous functions on $[0,\infty)$ such that $p(t) \geqslant 0$, $q(t) \geqslant 0$ and $h(y)$ is continuous on $(-\infty,\infty)$ subject to $h(y)y > 0$ for $y \neq 0$ while $\alpha > 0$ is the ratio of odd integers.

In recent years, third order homogeneous differential equations (linear and nonlinear) have been the main stream of investigations for many authors. Barrett [1], Bobrowski [2], Cecchi and Marini [3], Erbe ([4], [5]), Greguš ([7], [8]) are just some of them. They have given sufficient conditions for oscillation and nonoscillation and studied asymptotic behaviour of the solutions. The study of properties of nonlinear equations is not as extensive as for the linear case. Moreover, some results for linear cases fail in some nonlinear cases [12].

For the third order equation (1) much less work has been done. Motivation for the study of this equation comes from the works of Greguš, M. and Greguš Jr.M ([9], [10]) and Heidel [11] who studied qualitative behaviour of solutions of the special cases of (1), and the work of Parhi, N. and Parhi S. [14] who studied the nonhomogeneous equations with the remark that some of their results are not valid for the homogeneous case.

In the present paper, first we give sufficient conditions under which solutions of (1) are nonoscillatory. Secondly, we are concerned with asymptotic behaviour of nonoscillatory solutions.

We restrict our considerations to those real solutions y of (1) which exist on the half line $[T, \infty)$, where $T \geqslant 0$ depends on the particular solution, and are nontrivial in any neighbourhood of infinity. We may recall that a solution y of (1) on $[T, \infty)$ is said to be nonoscillatory if there exists a $t_1 \geqslant T$ such that $y(t) \neq 0$ for $t \geqslant t_1$; $y(t)$ is said to be oscillatory if for any $t_1 \geqslant T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$; it is said to be of Z-type if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

2.

In this section we give sufficient conditions under which solutions of (1) are nonoscillatory.

Theorem 2.1. *Suppose that $q(t) \leqslant p(t)$ for $t \in [0, \infty)$ and $q(t) \not\equiv 0$ on any subinterval of $[0, \infty)$. If y is a solution of (1) defined on $[T, \infty)$, $T \geqslant 0$, such that it satisfies $z'^{\alpha} + h(z) < 0$ on the subinterval of $[T, \infty)$ on which $y'(t) > 0$, while $z'^{\alpha} + h(z) > 0$ on the subinterval of $[T, \infty)$ on which $y'(t) < 0$, then y is nonoscillatory on $[T, \infty)$.*

P r o o f. Let y be a solution of (1) on $[T, \infty)$, $T \geqslant 0$. Multiplying (1) by $y'(t)$ we get

$$y'''(t)y'(t) = -q(t)y'^{\alpha}(t)y'(t) - p(t)h(y(t))y'(t) \tag{2}$$

Let us assume that $y'(t) > 0$ on some subinterval of $[T, \infty)$. Then the condition $y'^{\alpha} + h(y) < 0$ yields $h(y) < 0$, and so $y(t) < 0$. Thus we cannot have simultaneously

$y(t) > 0$ and $y'(t) > 0$. Therefore oscillatory or nonnegative Z-type solutions cannot exist, and in the proof it is sufficient to exclude nonpositive Z-type solutions.

Let y be a nonpositive Z-type solution with consecutive double zeros at a and $b (a < b)$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and hence $y'(t) > 0$ for $t \in (c, b)$. Now integrating (2) from c to b yields

$$\begin{aligned} 0 &= y''(t)y'(t) \mid_c^b \\ &\geqslant \int_c^b y''^2(t)\,\mathrm{d}t - \int_c^b q(t)\big[y'^{\alpha}(t) + h(y(t))\big]y'(t)\,\mathrm{d}t > 0, \end{aligned}$$

which is a contradiction.

When $y'(t) < 0$, the above argument can be repeated to complete the proof of the theorem. $\square$

Now we give an example to illustrate the above theorem.

Example 1.

$$y''' + \frac{1}{2}y'^3 + \frac{2\mathrm{e}^{3t} + \mathrm{e}^t}{2(\mathrm{e}^{3t} + \mathrm{e}^t)}(y + y^3) = 0, \qquad t \geqslant 0.$$

$y(t) = \mathrm{e}^{-t}$ and $y(t) = -\mathrm{e}^{-t}$ are nonoscillatory solutions of this equation.

Theorem 2.2. *Let* $1 \leqslant q(t)$. *If* y *is a solution of* (1) *on an interval on which* $y''^2(t) - y'^{\alpha+1}(t) < 0$ *holds, then* y *is nonoscillatory.*

Proof. Let $y(t)$ be of nonnegative Z-type with consecutive double zeros at a and $b (a < b)$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$. Integrating (2) from a to c, we get

$$\begin{aligned} 0 &= y''(t)y'(t) \mid_a^c \\ &\leqslant \int_a^c q(t)\big[y''^2(t) - y'^{\alpha+1}(t)\big]\,\mathrm{d}t - \int_a^c p(t)hy(y(t))y'(t)\,\mathrm{d}t. \end{aligned}$$

Since $h(y(t))$ is positive for $y(t) > 0$ we arrive at

$$0 \leqslant \int_a^c q(t)\big[y''^2(t) - y'^{\alpha+1}(t)\big]\,\mathrm{d}t - \int_a^c p(t)h(y(t))y'(t)\,\mathrm{d}t < 0,$$

which is a contradiction.

Next suppose that $y(t)$ is nonpositive Z-type with consecutive double zeros at a and $b (a < b)$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) < 0$ for

$t \in (a, c)$. Now integrating (2) from a to c, we obtain

$$\begin{aligned} 0 &= y''(t)y'(t)\,|_a^c \\ &\leqslant \int_a^c q(t)\big[y''^2(t) - y'^{\alpha+1}(t)\big]\,\mathrm{d}t - \int_a^c p(t)h(y(t))y'(t)\,\mathrm{d}t. \end{aligned}$$

As in the above case we get a contradiction.

Finally, to complete the proof, suppose that $y(t)$ is oscillatory. Let a, b and a' ($a < b < a'$) be any three consecutive zeros of $y(t)$ such that $y'(a) \leqslant 0$, $y'(b) \geqslant 0$, $y'(a') \leqslant 0$; so $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0 = y'(c')$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. We consider two cases, namely, $y''(b) \leqslant 0$ and $y''(b) > 0$. First, let $y''(b) \leqslant 0$. Integrating (2) between b and c', we obtain

$$\begin{aligned} 0 &\leqslant y''(t)y'(t)\,|_b^{c'} \\ &\leqslant \int_b^{c'} q(t)\big[y''^2(t) - y'^{\alpha+1}(t)\big]\,\mathrm{d}t - \int_b^{c'} p(t)h(y(t))y'(t)\,\mathrm{d}t \quad < 0, \end{aligned}$$

a contradiction. Hence $y''(b) > 0$. Since $y''(t)$ is continuous, $y''(t) > 0$ for $t \in [b, b + \delta_1)$, $0 < \delta_1 < c' - b$. So $y'(t)$ is increasing on $[b, b + \delta_1)$. Again $y'(c') = 0$ and $y'(t) > 0$ for $t \in (b, c')$ imply that $y'(t)$ is decreasing on $[c' - \delta_2, c']$, where $0 < \delta_2 < c' - b$. This in turn implies that $y''(t) < 0$ for $t \in [c' - \delta_2, c']$. Hence $y''(d) = 0$ for some $d \in (b, c')$. Integrating (2) from d to c' we get a contradiction again. □

Example 2.

$$y''' + t^{10/3}y'^{5/3} + \frac{t^4 + 6}{2t^4 + 4t^3 + 3t^2 + t}(y + y^3) = 0, \qquad t > 8.$$

Here $y(t) = 1 + 1/t$ is a nonoscillatory solution of the above equation satisfying the conditions of Theorem 2.2.

Theorem 2.3. *Any solution y of* (1) *satisfying the inequality*

$$z''^2 - q(t)z'^{\alpha+1} - p(t)h(z)z' > 0$$

on an interval on which $y'(t) > 0$, is nonoscillatory.

P r o o f. Let y be a solution of (1) satisfying conditions of the theorem. If possible, assume that y is of nonnegative Z-type. Let a and b ($a < b$) be any two consecutive double zeros of y. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and

$y'(t) > 0$ for $t \in (a, c)$. Now multiplying (1) by ${y'}^2(t)$ and integrating the resulting identity from a to c, we get

$$\begin{aligned} 0 &= y''(t){y'}^2(t) \mid_a^c \\ &= -\int_a^c q(t){y'}^\alpha(t){y'}^2(t)\,\mathrm{d}t - \int_a^c p(t)h(y(t)){y'}^2(t)\,\mathrm{d}t + 2\int_a^c y'(t)y''^2(t)\,\mathrm{d}t \\ &> 0, \end{aligned}$$

a contradiction.

Similarly we can show that $y(t)$ cannot be of nonpositive Z-type.

Suppose that $y(t)$ is oscillatory. Let a, b and a' $(a < b < a')$ be any three consecutive zeros of $y(t)$ such that $y'(a) \leqslant 0$, $y'(b) \geqslant 0$, $y'(a') \leqslant 0$; so $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. Thus there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0 = y'(c')$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. We consider two cases, namely, $y''(b) \leqslant 0$ and $y''(b) > 0$. Integrating (2) from b to c' in case $y''(b) > 0$ and from c to b in case $y''(b) \leqslant 0$, we get the required contradictions. □

Example 3.

$$y''' + \frac{1}{27}t^{-3}{y'}^3 + \frac{1}{4}t^{-6}y^3 = 0, \qquad t > 0.$$

This example illustrates the above theorem. Clearly $y(t) = t^{3/2}$ is a nonoscillatory solution of the equation.

3.

In this section we are concerned with the asymptotic behaviour of nonoscillatory solutions of (1). In [11], Heidel gave sufficient conditions under which a nonoscillatory solution of the equation (1) with $h(y) = y^r$ tends to zero as $t \to \infty$. In the following we extend some of Heidel's results to (1).

First we recall.

Lemma 3.1. *Consider*

$$z'' + q(t)z = 0 \tag{3}$$

where $q(t) \geqslant 0$. *If* z *is a nonoscillatory solution of* (3) *such that* $z(t) \neq 0$ *for* $t \in [a, \infty)$, $a > 0$, *and if* u *is a continuously differentiable function on* $[a, \infty)$ *such that* $u(b) = 0 = u(c)$, $a < b < c$, *and* $u(t) \not\equiv 0$ *on* $[b, c]$, *then*

$$\int_b^c [u'^2(t) - q(t)u^2(t)]\,\mathrm{d}t > 0.$$

For the proof of this lemma the reader is referred to [15]. □

Remark 3.1. $q(t)$ need not be nonnegative in Lemma 3.1. If $q(t) \leqslant 0$ then all solutions of (3) are nonoscillatory. For $q(t) \nleqslant 0$, sufficient conditions were given by Moore [16, p. 73], Winter [17] and Potter [16, p. 81] for nonoscillation of all solutions of (3).

Now we prove

Lemma 3.2. *Let $\int_0^\infty p(t)\,\mathrm{d}t = \infty$ and let $h(y)$ be nondecreasing. If y is a nonoscillatory solution of* (1) *and $y(t) \neq 0$ on $[t_0, \infty)$ then $|y|$ is not nondecreasing on $[t_0, \infty)$.*

Proof. Let $y(t) > 0$ and $y'(t) \geqslant 0$ for $t \geqslant t_0 > 0$. Integrating the inequality

$$y'''(t) \leqslant -p(t)h(y(t))$$

from t_0 to t, we get

$$y''(t) \leqslant y''(t_0) - h(y(t_0)) \int_{t_0}^t p(s)\,\mathrm{d}s.$$

This in turn implies that $y'(t) < 0$ for large t, which yields a contradiction.

If $y(t) < 0$ and $y'(t) \leqslant 0$ for $t \geqslant t_0 > 0$, proceeding as above we obtain a similar contradiction. □

Theorem 3.1. *Let $\alpha = 1$ in equation* (1). *Let $q(t) \leqslant M$, $M > 0$ and let the hypothesis of Lemma* 3.2 *hold. If y is a nonoscillatory solution of* (1) *such that $y(t)y'(t) < 0$ for $t \in [t_0, \infty)$, $t_0 \geqslant 0$, then $\lim\limits_{t\to\infty} y(t) = 0$.*

Proof. Let $y(t) > 0$ for $t \geqslant t_0$. So $y'(t) < 0$ for $t \geqslant t_0$ and hence $\lim\limits_{t\to\infty} y(t)$ exists. Let us assume $\lim\limits_{t\to\infty} y(t) > 0$. Integrating (1) from t_0 to t, we obtain

$$\begin{aligned} y''(t) &\leqslant y''(t_0) - M(y(t) - y(t_0)) - \int_{t_0}^t h(y(s))p(s)\,\mathrm{d}s \\ &\leqslant y''(t_0) - M(y(t) - y(t_0)) - h(y(t)) \int_{t_0}^t p(s)\,\mathrm{d}s. \end{aligned}$$

This yields that $y(t) < 0$ for large t, giving a contradiction.

Similarly, when $y(t) < 0$, $y'(t) > 0$ for $t \geqslant t_0$ and $\lim\limits_{t\to\infty} y(t) < 0$ we get a contradiction.

This completes the proof of the theorem. □

In the above proof; the assumption employed is weaker then in Heidel [11, Theorem 3.7], if $t_0 > 1$.

Theorem 3.2. *Let $\alpha = 1$ in equation* (1). *Let the hypothesis of Lemma* 3.2 *hold and let equation* (3) *be nonoscillatory. If y is a nonoscillatory solution of* (1) *and $y(t) \neq 0$ on $[t_0, \infty)$ $t_0 \geqslant 0$, then $y(t)y'(t) < 0$ for all $t \in [t_0, \infty)$.*

Proof. Let $y(t) < 0$ for $t \geqslant t_0$. From Lemma 3.2 it follows that $y'(t) \nleqslant 0$. Assume that $y'(t)$ is oscillatory or nonnegative Z-type with consecutive zeros at a and b ($t_0 < a < b$) such that $y'(t) > 0$ for $t \in (a, b)$. Integrating (2) from a to b, we get

$$\begin{aligned} 0 &> y''(t)y'(t)\mid_a^b + \int_a^b p(t)h(y(t))y'(t)\,\mathrm{d}t \\ &= \int_a^b \left[y''^2(t) - q(t){y'}^2(t)\right]\mathrm{d}t \end{aligned}$$

and by Lemma 3.1 we arrive at a contradiction. So $y'(t) > 0$.

Let $y(t) > 0$ for $t \geqslant t_0$. It is clear from Lemma 3.2 that $y'(t) \ngeqslant 0$. Let $y'(t)$ be oscillatory or nonpositive Z-type. Let a and b ($t_0 < a < b$) be consecutive zeros of $y'(t)$ such that $y'(t) < 0$ for $t \in (a, b)$. Integrating (2) from a to b, we obtain

$$\begin{aligned} 0 &> \int_a^b p(t)h(y(t))y'(t)\,\mathrm{d}t \\ &= \int_a^b \left[y''^2(t) - q(t){y'}^2(t)\right]\mathrm{d}t \\ &> 0, \end{aligned}$$

which is a contradiction. So $y'(t) < 0$.

Hence the theorem. □

Example 4.

$$y''' + \frac{1}{4t^2}y' + \frac{4t^2\beta^3 + \beta}{4t^2(1 + \mathrm{e}^{-2\beta t})}(y + y^3) = 0, \quad t > 1, \beta > 0 \text{ is a constant.}$$

Note that this example illustrates Lemma 3.2, Theorem 3.1 and Theorem 3.2. Clearly $y(t) = \mathrm{e}^{-\beta t}$ is a positive solution of this equation. On the other hand $z'' + \frac{1}{4t^2}z = 0$ $t \geqslant 1$ is nonoscillatory because $z(t) = \sqrt{t}\ln t$ is a nonoscillatory solution of the equation.

In the following an attempt has been made to remove the restriction on α in Theorem 3.1.

Theorem 3.3. *Let the suppositions of Theorem* 2.1 *and Lemam* 3.3 *hold. If y is a solution of* (1) *having the properties as in Theorem* 2.1, *then* $\lim\limits_{t\to\infty} y(t) = 0$.

Proof. Let $y(t) > 0$ for $t \geqslant t_0 > 0$. From Lemma 3.2 it follows that $y'(t) \ngeqslant 0$. Let $y'(t)$ be oscillatory or nonpositive Z-type with consecutive zeros at a and b $(t_0 < a < b)$ such that $y'(t) < 0$ for $t \in (a, b)$. Now integration of (2) from a to b yields

$$\begin{aligned} 0 &= y''(t)y'(t) \mid_a^b \\ &\geqslant -\int_a^b q(t)\big[y'^{\alpha}(t) + h(y(t))\big]y'(t)\,\mathrm{d}t \\ &> 0, \end{aligned}$$

a contradiction. So $y'(t) < 0$. Consequently, $\lim\limits_{t\to\infty} y(t)$ exists. Now let us assume $\lim\limits_{t\to\infty} y(t) > 0$. From (1) it follows that $y''(t)$ is monotonic decreasing. We claim that there exists a $t_1 \geqslant t_0$ such that $y''(t) > 0$ for $t \geqslant t_1$. If not, for every $t_1 \geqslant t_0$ there exists a $t_2 \geqslant t_1$ such that $y''(t_2) \leqslant 0$. So $t \geqslant t_2$ implies that $y''(t) \leqslant 0$. Hence $y(t) < 0$ for large t. This is a contradicition. So our claim holds. Now integrating (1) from t_1 to t, we get

$$\begin{aligned} y''(t) &= y''(t_1) - \int_{t_1}^t q(s)y'^{\alpha}(s)\,\mathrm{d}s - \int_{t_1}^t p(s)h(y(s))\,\mathrm{d}s \\ &\leqslant y''(t_1) - y'^{\alpha}(t_1)\int_{t_1}^t q(s)\,\mathrm{d}s - h(y(t))\int_{t_1}^t p(s)\,\mathrm{d}s, \end{aligned}$$

which in turn implies $y''(t) < 0$ for large t, contradicting $y''(t) > 0$. Hence $\lim\limits_{t\to\infty} y(t) = 0$.

When $y(t) < 0$ for $t \geqslant t_0 > 0$, the above argument can be repeated to complete the proof of the theorem. □

Remark 3.2. The results obtained here can be extended to the equation

$$(r(t)y'')' + q(t)y'^{\alpha} + p(t)h(y) = 0$$

where r is a continuous and positive function. The only assumption needed for minor manipulations is

$$\int_0^\infty \frac{\mathrm{d}t}{r(t)} = \infty$$

to hold, whenever necessary.

Acknowledgment. The authors would like to thank the referee for his valuable suggestions and useful comments.

References

[1] *Barrett, J.H.*: Oscillation theory of ordinary linear differential equations. Advances in Math. *3* (1969), 415–509.

[2] *Bobrowski, D.*: Asymptotic behaviour of functionally bounded solutions of the third order nonlinear differential equation. Fasc. Math. (Poznañ) *10* (1978), 67–76.

[3] *Cecchi, M. and Marini, M.*: On the oscillatory behaviour of a third order nonlinear differential equation. Nonlinear Anal. *15* (1990), 141–153.

[4] *Erbe, L. H.*: Oscillation, nonoscillation and asymptotic behaviour for third order nonlinear differential equation. Ann. Math. Pura Appl. *110* (1976), 373–393.

[5] *Erbe, L. H. and Rao, V. S. M.*: Nonoscillation results for third order nonlinear differential equations. J. Math. Analysis Applic. *125* (1987), 471–482.

[6] *Greguš, M.*: Third Order Linear Differential Equations. D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, 1987.

[7] *Greguš, M.*: On the asymptotic properties of solutions of nonlinear third order differential equation. Archivum Mathematicum (Brno) *26* (1990), 101–106.

[8] *Greguš, M.*: On the oscillatory behaviour of certain third order nonlinear differential equation. Archivum Mathematicum (Brno) *28* (1992), 221–228.

[9] *Greguš, M. and Greguš Jr. M.*: On the oscillatory properties of solutions of a certain nonlinear third order differential equation. J. Math. Analysis Applic. *181* (1994), 575–585.

[10] *Greguš, M. and Greguš Jr., M.*: Asymptotic properties of solution of a certain nonautonomous nonlinear differential equations of the third order. Bollettino U.M.I. *(7) 7-A* (1993), 341–350.

[11] *Heidel, J. W.*: Qualitative behaviour of solution of a third order nonlinear differential equation. Pacific J. Math. *27* (1968), 507–526.

[12] *Heidel J. W.*: The existence of oscillatory solution for a nonlinear odd order nonlinear differential equation. Czechoslov. Math. J. *20* (1970), 93–97.

[13] *Ladde, G. S., Lakshmikantham, V. and Zhank, B. G.*: Oscillation Theory of Differential Equations with Deviating Arguments. Marchel Dekker, Inc., New York, 1987.

[14] *Parhi, N. and Parhi, S.*: Nonoscillation and asymptotic behaviour forced nonlinear third order differential equations. Bull. Inst. Math. Acad. Sinica *13* (1985), 367–384.

[15] *Parhi, N. and Parhi, S.*: On the behaviour of solution of the differential equations $(r(t)y'')' + q(t)(y')^\beta + p(t)y^\alpha = f(t)$. Annales Polon. Math. *47* (1986), 137–148.

[16] *Swanson, C.A.*: Comparison and Oscillation Theory of Linear Differential Equations. New York and London, Acad. Press, 1968.

[17] *Wintner, A.*: On the nonexistence of conjugate points. Amer. J. Math. *73* (1951), 368–380.

Authors' addresses: Aydın Tiryaki, Hacettepe University, Faculty of Science, Mathematics Department, 06532 Beytepe-Ankara, Turkey; A. Okay Çelebi, Middle East Technical University, Mathematics Department, 06531 Ankara, Turkey.