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ON L-FUZZY IDEALS IN SEMIRINGS I

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Abstract. In this paper we extend the concept of an L-fuzzy (characteristic) left (resp. right) ideal of a ring to a semiring R, and we show that each level left (resp. right) ideal of an L-fuzzy left (resp. right) ideal μ of R is characteristic iff μ is L-fuzzy characteristic.

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Following the introduction of fuzzy sets by L. A. Zadeh ([9]), the fuzzy set theory developed by Zadeh himself and others can be found in mathematics and many applied areas. In 1982, W. Liu ([5]) defined and studied fuzzy subrings as well as fuzzy ideals in rings. Subsequently, T. K. Mukherjee and M. K. Sen ([6]), K. L. N. Swamy and U. M. Swamy ([7]), and Zhang Yue ([8]) fuzzified certain standard concepts/results on rings and ideals. The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (e.g., [1, 2, 3, 4]). In this paper we extend the concept of an L-fuzzy (characteristic) left (resp. right) ideal of a ring to a semiring R, and we show that each level left (resp. right) ideal of L-fuzzy left (resp. right) ideal μ of R is characteristic iff μ is L-fuzzy characteristic. It should be noted that usually the transiton from rings to semirings is a delicate matter requiring careful adjustment of definitions and results in order to succeed.

By a *semiring* we shall mean a set R endowed with two associative binary operations called *addition* and *multiplication* (denoted by $+$ and $\cdot$, respectively) satisfying the following conditions:

(i) addition is a commutative operation,

(ii) there exists $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$,

(iii) multiplication distributes over addition both from the left and from the right. From now on we write R and S for semirings. A non-empty subset A of R is a *left* (resp. *right*) *ideal* if $x, y \in A$ and $r \in R$ imply that $x+y \in A$ and $rx \in A$ (resp. $xr \in A$). If A is both left and right ideal of R, we say A is a two-sided ideal, or simply, ideal of R. A mapping $f\colon R \to S$ is called a *homomorphism* if $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$. We note that if $f\colon R \to S$ is an onto homomorphism and if A is a left (resp. right) ideal of R, then $f(A)$ is a left (resp. right) ideal of S.

Throughout this paper $L = (L, \leqslant, \wedge, \vee)$ will be a completely distributive lattice, which has the least and the greatest elements, say 0 and 1, respectively. Let X be a non-empty (usual) set. An *L-fuzzy set* in X is a map $\mu\colon X \to L$, and $\mathscr{F}(X)$ will denote the set of all L-fuzzy sets in X. If $\mu, \nu \in \mathscr{F}(X)$, then $\mu \subseteq \nu$ if and only if $\mu(x) \leqslant \nu(x)$ for all $x \in X$, and $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$. It is easily seen that $\mathscr{F}(X) = (\mathscr{F}(X), \subseteq, \wedge, \vee)$ is a completely distributive lattice, which has the least and the greatest elements, say $\mathbf{0}$ and $\mathbf{1}$, respectively in natural manner, where $\mathbf{0}(x) = 0$, $\mathbf{1}(x) = 1$ for all $x \in X$.

Given any two sets X and X', let $\mu \in \mathscr{F}(X)$ and let $f\colon X \to X'$ be any function. We define $\nu \in \mathscr{F}(X')$ by

$$\nu(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if} f^{-1}(y) \neq \emptyset, y \in X', \\ 0 & \text{otherwise,} \end{cases}$$

and we call ν the *image* of μ under f, written $f(\mu)$. For any $\nu \in \mathscr{F}(f(X))$, we define $\mu \in \mathscr{F}(X)$ by $\mu(x) = \nu(f(x))$ for all $x \in X$, and we call μ the *preimage* of ν under f which is denoted by $f^{-1}(\nu)$.

Definition 1. An L-fuzzy set $\mu(\in \mathscr{F}(R))$ is called an *L-fuzzy left* (resp. *right*) *ideal* of R if for all $x, y \in R$,

(i) $\mu(x+y) \geqslant \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(xy) \geqslant \mu(y)$ (resp. $\mu(xy) \geqslant \mu(x)$).

An L-fuzzy set μ is an *L-fuzzy ideal* of R if and only if it is both L-fuzzy left and right ideal of R. It follows from the definition of the semiring that if μ is an L-fuzzy left (resp. right) ideal of R, then $\mu(0) \geqslant \mu(x)$ for all $x \in X$. As the idea of a semiring is a generalization of the idea of a ring, the notion of L-fuzzy left (resp. right) ideal of a semiring is also a generalization of the notion of L-fuzzy left (resp. right) ideal in rings. Hence, every L-fuzzy left (resp. right) ideal of a ring is L-fuzzy left (resp. right) ideal of a semiring. But the converse need not at all be true. Consider the following example.

Example 2. (a). Let $R := \{0,1,2,3\}$ be a set with two associative binary operations:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

Then we can easily see that $(R;+,\cdot)$ is a semiring. Define an L-fuzzy set $\mu\colon\ R \to L$ by $\mu(3) < \mu(2) < \mu(1) < \mu(0)$. Then μ is an L-fuzzy left ideal of the semiring R, but μ is not an L-fuzzy left (ring-) ideal of R, since $\mu(x-y)$ is not defined for any $x, y \in R$.

(b). The semiring of non-negative real numbers with respect to addition and multiplication is of great practical importance and yet is not a ring, nor can both operations be transformed simultaneously to obtain a ring. For this semiring there are many L-fuzzy ideals of natural interest. (E.g., with respect to the study of the exponential distribution in probability theory for example.)

Proposition 3. *Let $\mu \in \mathscr{F}(R)$. Then μ is an L-fuzzy left (resp. right) ideal of R if and only if, for any $t \in L$ such that $\mu_t \neq \emptyset$, μ_t is a left (resp. right) ideal of R, where $\mu_t = \{x \in R \mid \mu(x) \geqslant t\}$, which is called a level subset of μ.*

Proof. If μ is an L-fuzzy left (resp. right) ideal of R, it is easy to see that $\mu_t \neq \emptyset$ is a left (resp. right) ideal of R. Conversely, let all $\mu_t \neq \emptyset$ be left (resp. right) ideal of R. Then for all $x, y \in R$, we have $x, y \in \mu_{\min\{\mu(x),\mu(y)\}}$, so $x + y \in \mu_{\min\{\mu(x),\mu(y)\}}$. Thus $\mu(x+y) \geqslant \min\{\mu(x),\mu(y)\}$. Noticing that $x \in \mu_{\mu(x)}$, we obtain $rx \in \mu_{\mu(x)}$ (resp. $xr \in \mu_{\mu(x)}$) for all $r \in R$. It follows that $\mu(rx) \geqslant \mu(x)$ (resp. $\mu(xr) \geqslant \mu(x)$). Therefore μ is an L-fuzzy left (resp. right) ideal of R. $\square$

If μ is an L-fuzzy left (resp. right) ideal of R, we call μ_t $(\neq \emptyset)$ a *level left* (resp. *right*) *ideal* of μ. If $\mu \in \mathscr{F}(R)$ is an L-fuzzy left (resp. right) ideal of R, then the set $R_\mu = \{x \in R \mid \mu(x) \geqslant \mu(0)\}$ is a left (resp. right) ideal of R.

Theorem 4. *Let A be any left (resp. right) ideal of R. Then there exists an L-fuzzy left (resp. right) ideal μ of R such that $\mu_t = A$ for some $t \in L$.*

Proof. If we define a L-fuzzy set in R by

$$\mu(x) = \begin{cases} t & \text{if} x \in A, \\ 0 & \text{otherwise} \end{cases}$$

for some $t \in L$, then it follows that $\mu_t = A$. For given $s \in L$ we have

$$\mu_s = \begin{cases} \mu_0(= R) & \text{if} s = 0, \\ \mu_t(= A) & \text{if} s \leqslant t, \\ \emptyset & \text{if} t < s \leqslant 1. \end{cases}$$

Since A and R itself are left (resp. right) ideals of R, it follows that every non-empty level subset μ_s of μ is a left (resp. right) ideal of R. By Proposition 3, μ is an L-fuzzy left (resp. right) ideal of R, which satisfies the conditions of the theorem. $\square$

Theorem 5. *Let $\mu \in \mathscr{F}(R)$ be an L-fuzzy left (resp. right) ideal of R. Then two level left (resp. right) ideals μ_s, μ_t (with $s < t$ in L) of μ are equal if and only if there is no $x \in R$ such that $s \leqslant \mu(x) < t$.*

Proof. Suppose $s < t$ in L and $\mu_s = \mu_t$. If there exists a $x \in R$ such that $s \leqslant \mu(x) < t$, then μ_t is a proper subset of μ_s, a contradiction. Conversely, suppose that there is no $x \in R$ such that $s \leqslant \mu(x) < t$. Note that $s < t$ implies $\mu_t \subseteq \mu_s$. If $x \in \mu_s$, then $\mu(x) \geqslant s$, and so $\mu(x) \geqslant t$ because $\mu(x) \nless t$. Hence $x \in \mu_t$, and $\mu_s = \mu_t$. This completes the proof. $\square$

For any $\mu \in \mathscr{F}(R)$ we denote by $\mathrm{Im}(\mu)$ the image set of μ.

Theorem 6. *Let $\mu \in \mathscr{F}(R)$ be an L-fuzzy left (resp. right) ideal of R. If* $\mathrm{Im}(\mu)= \{t_1, t_2, \ldots, t_n\}$, *where $t_1 < t_2 < \ldots < t_n$, then the family of left (resp. right) ideals μ_{t_i} $(i = 1, \ldots, n)$ constitutes the collection of all level left (resp. right) ideals of μ.*

Proof. If $t \in L$ with $t < t_1$, then $\mu_{t_1} \subseteq \mu_t$. Since $\mu_{t_1} = R$, we have $\mu_t = R$ and $\mu_t = \mu_{t_1}$. If $t \in L$ with $t_i < t < t_{i+1}$ $(1 \leqslant i \leqslant n-1)$, then there is no $x \in R$ such that $t \leqslant \mu(x) < t_{i+1}$. It follows from Theorem 5 that $\mu_t = \mu_{t_{i+1}}$. This shows that for any $t \in L$ with $t \leqslant \mu(0)$, the level left (resp. right) ideal μ_t is in $\{\mu_{t_i} \mid 1 \leqslant i \leqslant n\}$. This completes the proof. $\square$

Theorem 7. *An onto homomorphic preimage of an L-fuzzy left (resp. right) ideal is an L-fuzzy left (resp. right) ideal.*

Proof. Let $f\colon R \to S$ be an onto homomorphism. Let $\nu \in \mathscr{F}(S)$ be an L-fuzzy left ideal and let μ be the preimage of ν under f. Then for any $x, y \in R$,

$$\begin{aligned} \mu(x+y) &= \nu(f(x+y)) \\ &= \nu(f(x) + f(y)) \\ &\geqslant \min\{\nu(f(x)), \nu(f(y))\} \\ &= \min\{\mu(x), \mu(y)\} \end{aligned}$$

and $\mu(xy) = \nu(f(xy)) = \nu(f(x)f(y)) \geqslant \nu(f(y)) = \mu(y)$. This shows that μ is an L-fuzzy left ideal of R. The other cases are similar. □

Proposition 8. *Let f be a mapping from a set X to a set Y, and let $\mu \in \mathscr{F}(X)$. Then for every $t \in L$, $t \neq 0$,*

$$(f(\mu))_t = \bigcap_{0<s<t} f(\mu_{t-s}).$$

P r o o f. Let $t \in L$, $t \neq 0$. If $y \in (f(\mu))_t$, then

$$t \leqslant (f(\mu))(y) = \sup_{z \in f^{-1}(y)} \mu(z).$$

This means that there exists $x_0 \in f^{-1}(y)$ such that $\mu(x_0) > t - s$ for every $s \in L$ with $0 < s < t$, and so $y = f(x_0) \in f(\mu_{t-s})$. Therefore $y \in \bigcap\limits_{0<s<t} f(\mu_{t-s})$. Conversely, let $y \in \bigcap\limits_{0<s<t} f(\mu_{t-s})$. Then $y \in f(\mu_{t-s})$ for every $s \in L$ with $0 < s < t$, which implies that there exists $x_0 \in \mu_{t-s}$ such that $y = f(x_0)$. It follows that $\mu(x_0) \geqslant t - s$ and $x_0 \in f^{-1}(y)$, so that

$$(f(\mu))(y) = \sup_{z \in f^{-1}(y)} \mu(z) \geqslant \sup_{0<s<t} \{t - s\} = t.$$

Hence $y \in (f(\mu))_t$, and we complete the proof. □

Theorem 9. *Let $f\colon R \to S$ be an onto homomorphism and let μ be an L-fuzzy left (resp. right) ideal of R. Then the homomorphic image $f(\mu)$ of μ under f is an L-fuzzy left (resp. right) ideal of S.*

P r o o f. In view of Proposition 3 it is sufficient to show that each non-empty level subset of $f(\mu)$ is a left (resp. right) ideal of S. Let $(f(\mu))_t$ be a non-empty level subset of $f(\mu)$ for every $t \in L$. If $t = 0$ then $(f(\mu))_t = S$. Assume $t \neq 0$. By Proposition 8, $(f(\mu))_t = \bigcap\limits_{0<s<t} f(\mu_{t-s})$. Hence $f(\mu_{t-s})$ is non-empty for each $0 < s < t$, and so μ_{t-s} is a nonempty level subset of μ for every $0 < s < t$. Since μ is an L-fuzzy left (resp. right) ideal of R, it follows from Proposition 3 that μ_{t-s} is a left (resp. right) ideal of R. Since f is an onto homomorphism, $f(\mu_{t-s})$ is a left (resp. right) ideal of S. Hence $(f(\mu))_t$ being an intersection of a family of left (resp. right) ideals is also a left (resp. right) ideal of S. The proof is complete. □

Definition 10. A left (resp. right) ideal A of R is said to be *characteristic* if $f(A) = A$ for all $f \in \mathrm{Aut}(R)$, where $\mathrm{Aut}(R)$ is the set of all automorphisms of R. An L-fuzzy left (resp. right) ideal μ of R is said to be *L-fuzzy characteristic* if $\mu(f(x)) = \mu(x)$ for all $x \in R$ and $f \in \mathrm{Aut}(R)$.

Theorem 11. *Let μ be an L-fuzzy left (resp. right) ideal of R and let $f\colon R \to R$ be an onto homomorphism. Then the mapping $\mu^f \in \mathscr{F}(R)$, defined by $\mu^f(x) = \mu(f(x))$ for all $x \in R$, is an L-fuzzy left (resp. right) ideal of R.*

P r o o f. For any $x, y \in R$, we have

$$\begin{aligned} \mu^f(x+y) &= \mu(f(x+y)) \\ &= \mu(f(x)+f(y)) \\ &\geqslant \min\{\mu(f(x)), \mu(f(y))\} \\ &= \min\{\mu^f(x), \mu^f(y)\} \end{aligned}$$

and

$$\begin{aligned} \mu^f(xy) &= \mu(f(xy)) \\ &= \mu(f(x)f(y)) \\ &\geqslant \mu(f(y)) \text{ (resp. } \mu(f(x))) \\ &= \mu^f(y) \text{ (resp. } \mu^f(x)). \end{aligned}$$

Hence μ^f is an L-fuzzy left (resp. right) ideal of R. □

Theorem 12. *If μ is an L-fuzzy characteristic left (resp. right) ideal of R, then each level left (resp. right) ideal of μ is characteristic.*

P r o o f. Let μ be an L-fuzzy characteristic left (resp. right) ideal of R and let $f \in \mathrm{Aut}(R)$. For any $t \in L$, if $y \in f(\mu_t)$, then $\mu(y) = \mu(f(x)) = \mu(x) \geqslant t$ for some $x \in \mu_t$ with $y = f(x)$. It follows that $y \in \mu_t$. Conversely, if $y \in \mu_t$, then $t \leqslant \mu(y) = \mu(f(x)) = \mu(x)$ for some $x \in R$ with $y = f(x)$. It follows that $y \in f(\mu_t)$. □

To prove the converse of Theorem 12, we need the following lemma.

Lemma 13. *Let μ be an L-fuzzy left (resp. right) ideal of R and let $x \in R$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.*

P r o o f. Straightforward. □

Theorem 14. *Let μ be an L-fuzzy left (resp. right) ideal of R. If each level left (resp. right) ideal of μ is characteristic, then μ is L-fuzzy characteristic.*

P r o o f. Let $x \in R$ and $f \in \mathrm{Aut}(R)$. If $\mu(x) = t \in L$, then by Lemma 13 $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$. Since each level left (resp. right) ideal of μ is characteristic, $f(x) \in f(\mu_t) = \mu_t$. Assume $\mu(f(x)) = s > t$. Then $f(x) \in \mu_s = f(\mu_s)$. Since f is one-to-one, it follows that $x \in \mu_s$, a contradiction. Hence $\mu(f(x)) = t = \mu(x)$, showing that μ is L-fuzzy characteristic. □

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