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σ-ELEMENTS IN MULTIPLICATIVE LATTICES

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All rings are assumed commutative with identity. By a multiplicative lattice, we mean a complete lattice L, with least element 0 and compact greatest element 1, on which there is defined a commutative, associative, completely join distributive product for which 1 is a multiplicative identity. By a C-lattice, we mean a multiplicative lattice which is generated under joins by a multiplicatively closed subset of compact elements. It is easy to see that in a C-lattice L, the set L_* of compact elements is multiplicatively closed. Throughout we assume that L is a C-lattice

An element $p < 1$ in L is said to be *prime* if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$. If 0 is prime, L is said to be a *domain*. By a *filter* on L_* we mean a multiplicatively closed subset $F \subseteq L_*$ such that $a \in F$, $b \in L$ and $a \leqslant b$ imply $b \in F$. We use $\mathfrak{F}(L_*)$ to denote the set of all filters of L_*. For any $a \in L_*$, the smallest filter containing a is denoted by $[a)$, so $[a) = \{x \in L_* \mid x \geqslant a^n$ for some nonnegative integer $n\}$. For any $a \in L$ and any $F \in \mathfrak{F}(L_*)$, we define $a_F = \bigvee\{x \in L_* \mid xy \leqslant a$ for some $y \in F\}$, and $L_F = \{a_F \mid a \in L\}$. For any prime element p of L, we define $F_p = \{x \in L_* \mid x \nleqslant p\}$, so $F_p \in \mathfrak{F}(L_*)$. In this case we denote $L_{(F_p)}$ by L_p and for $a \in L$, $a_p = a_{(F_p)}$. An element $m < 1$ in L is said to be *maximal* if $m < x \leqslant 1$ implies $x = 1$.It is easily seen that maximal elements are prime. For any filter F on L_*, L_F is again a multiplicative lattice under the same order as L with multiplication defined by $ab = (ab)_F$, where the right side is computed in L.

An element $a \in L$ is *nilpotent* if $a^n = 0$ for some positive integer n. The lattice L is said to be *reduced* if 0 is the only nilpotent element of L. We say that an element a has a property *locally* if a_m has the property in L_m for every maximal element m. For example, we say that an element $a \in L$ is locally nilpotent if a_m is nilpotent in L_m for every maximal element m.

We denote the residual of a by b by $a : b$. In a C-lattice, we have $a : b = \bigvee\{x \in L_* \mid xb \leqslant a\}$. The lattice L is said to be *quasiregular* if for any $x \in L_*$, there exists $y \in L_*$ such that $(0 : (0 : x)) = (0 : y)$. An element $a \in L$ is said to be *complemented* if it satisfies $ab = 0$ and $a \vee b = 1$, for some b. The lattice L is said to be a *regular* lattice if every compact element $a \in L$ is complemented. L is a *Baer lattice* if, for all $x \in L_*$, $(0 : (0 : x)) \vee (0 : x) = 1$. L is said to be *M-normal* if every prime element contains a unique minimal prime element. For various characterizations of quasiregular lattices, regular lattices, Baer lattices and M-normal lattices, the reader is referred to [5] and [6].

An element a of L is a *$*$-element* if $a = 0_F$ for some $F \in \mathfrak{F}(L_*)$. The element a is said to be a *Baer element* if for any $x \in L_*$, $x \leqslant a$ implies $(0 : (0 : x)) \leqslant a$. Baer elements and $*$-elements have been used to characterize quasiregular lattices, M-normal lattices and Baer lattices (see [6]).

The reader is referred to [4], for general background and terminology.

We begin with the following definitions.

Definition 1. An element $a \in L$ is a σ-element if, for every compact element $x \leqslant a$, $a \vee (0 : x) = 1$.

Definition 2. $\sigma(L) = \{a \in L \mid a \text{ is a } \sigma\text{-element}\}$.

It can be easily verified that $\sigma(L)$ is closed under finite meets, finite products and arbitrary joins. Also $0,\, 1 \in \sigma(L)$. Hence $\sigma(L)$ is a multiplicative lattice under the same order as L. A σ-element $a \in L$ is said to be a prime σ-element if a is prime in $\sigma(L)$. An σ-element $a \in L$ is said to be a maximal σ-element if a is maximal in $\sigma(L)$. Every maximal σ-element is a prime σ-element, and every σ-element is contained in a maximal σ-element.

Note that a compact element is a σ-element if and only if it is a complemented element. The following gives additional characterizations of σ-elements.

Proposition 1. *The following statements are equivalent for an element $a \in L$:*

(i) *a is locally complemented.*

(ii) *a is a σ-element.*

(iii) *$a = \bigwedge\{0_m \mid m$ is a maximal element containing $a\}$.*

P r o o f. (i) $\Rightarrow$ (ii). Suppose (i) holds. Assume $x \in L_*$ and $x \leqslant a$. Suppose $a \vee (0 : x) \neq 1$. Then $a \vee (0 : x) \leqslant m$ for some maximal element m of L. Note that the only complemented elements of L_m are 0_m and 1. Then $a_m \leqslant m_m$, and so by (1), $a_m = 0_m$. It follows that $(0 : x)_m = (0_m : x_m) = 1 \nleqslant m_m = m$, which contradicts the choice of m. Therefore a is a σ-element.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let m be a maximal element such that $a \leqslant m$. Then, for any compact element $x \leqslant a$, $(0 : x) \nleqslant m$ and $x(0 : x) = 0$. As L is a C-lattice, it follows that $x \leqslant 0_m$, and hence that $a \leqslant 0_m$. Therefore $a \leqslant \bigwedge\{0_m \mid m$ is a maximal element containing $a\}$. If y is compact and $y \leqslant \bigwedge\{0_m \mid m$ is a maximal element containing $a\}$, and if p is any maximal element, then $0_p : y_p = 1_p$ if $a \leqslant p$, and $a_p = 1_p$ if $a \nleqslant p$. Hence, $(a \vee (0 : y))_p = 1_p$ for every maximal element p, so $a \vee (0 : y) = 1$. Then $y = ay \leqslant a$, so (iii) holds. The implication (iii) $\Rightarrow$ (i) is obvious. □

Remark 1. By Proposition 1, every σ-element is the meet of $*$-elements.

It is convenient to record the following for later reference.

Proposition 2. *The following are equivalent for a prime element $p \in L$.*
(i) *p is a minimal prime over $a \in L$.*
(ii) *For any $x \in L_*$, $x \leqslant p$ implies there exists $y \nleqslant p$ such that $x^n y \leqslant a$ for some positive integer n.*

P r o o f. This is given by Lemma 3.5 of [3]. □

We now characterize M-normal lattices in terms of σ-elements.

Theorem 1. *Let L be reduced. Then the following statements are equivalent:*
(i) *Each maximal element contains a unique minimal prime element.*
(ii) *For every maximal element m of L, L_m is a domain.*
(iii) *L is M-normal.*
(iv) *Every $*$-element is a σ-element.*
(v) *Every minimal prime element is a σ-element.*
(vi) *Every minimal prime element is a maximal σ-element.*

P r o o f. (i) $\Rightarrow$ (ii). Suppose (i) holds. Let m be a maximal element of L. Then $0_m = 0_{F_m}$ is a $*$-element, so by Lemma 6 of [6], 0_m is the meet of all minimal prime elements containing it. By (i) 0_m is a prime element and so (ii) holds.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let p be a prime element. Then $p \leqslant m$ for some maximal element m of L. Then $0_m \leqslant p$ and 0_m is the only minimal prime element contained in p. Therefore L is M-normal.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds. Let a be a $*$-element. Then $a = 0_F$ for some $F \in \mathfrak{F}(L_*)$. Let $x \leqslant a$ be any compact element. Then $xy = 0$ for some $y \in F$. By (iii) and by Theorem 7 of [6], $(0 : x) \vee (0 : y) = 1$. Since $y \in F$, $(0 : y) \leqslant 0_F = a$, so $a \vee (0 : x) = 1$ and hence a is a σ-element.

(iv) $\Rightarrow$ (i). Suppose p_1 and p_2 are two distinct minimal prime elements. Choose any compact element $x \leqslant p_1$ such that $x \nleqslant p_2$. It follows from Proposition 2 that $xy = 0$ for some compact element $y \nleqslant p_1$. As $(0 : x) = 0_{[x)}$, $(0 : x)$ is a $*$-element, so by (iv), $(0 : x)$ is a σ-element and hence $(0 : x) \vee (0 : y) = 1$. Since $(0 : x) \leqslant p_2$ and $(0 : y) \leqslant p_1$, it follows that $p_1 \vee p_2 = 1$ and hence every maximal element contains a unique minimal prime element.

(iv) $\Rightarrow$ (v). Assume (iv). Let p be a minimal prime of L. It follows from Proposition 2 that $p = 0_p$. Hence, p is a σ-element by (iv).

(v) $\Rightarrow$ (vi). Assume (v) holds. Let p be a minimal prime element and assume $p \leqslant a \leqslant m$ for some σ-element a and some maximal element m of L. By Proposition 1, a is locally complemented, so $p = p_m = a_m = 0_m$ and therefore $a \leqslant a_m \leqslant p_m = p$. Hence (vi) holds.

(vi) $\Rightarrow$ (i). Assume (vi). Let m be a maximal element and let $p \leqslant m$ be a minimal prime element. By Proposition 1, p is locally complemented, so $p = 0_m$, and hence p is the only minimal prime $\leqslant m$. □

It can be easily shown that an ideal I of a ring R is a pure ideal ($x \in I$ implies $xy = x$ for some $y \in I$) if and only if I is a σ-ideal (see [2] and [7]). Pure ideals have been studied extensively in [1], [2] and [7] and σ-ideals have been studied by Cornish [9] in the case of distributive lattices. The following characterizes reduced Baer lattices in terms of σ-elements.

Theorem 2. *Suppose L is reduced. Then L is a Baer lattice if and only if every Baer element is a σ-element.*

Proof. Suppose L is a Baer lattice. Then by Theorem 10 of [6], L is M-normal and quasiregular. As L is quasiregular, by Theorem 2 of [6], every Baer element is a $*$-element. It follows from Theorem 1 that every Baer element is a σ-element.

Conversely, assume every Baer element is a σ-element and $x \in L_*$. It is observed in [3](page 63) that $(0 : (0 : x))$ is a Baer element. As $x \leqslant (0 : (0 : x))$, by hypothesis $(0 : (0 : x)) \vee (0 : x) = 1$ and hence L is a Baer lattice. □

Regular lattices can also be characterized in terms of σ-elements.

Theorem 3. *L is regular if and only if every element is a σ-element.*

Proof. If every element is a σ-element, then $x \vee (0 : x) = 1$ for every $x \in L_*$, and so L is regular.

Conversely, assume that L is regular. Then every compact element is complemented. Note that every complemented element is a σ-element. So every compact element is a σ-element. As L is compactly generated and the arbitrary join of σ-elements is a σ-element, it follows that every element is a σ-element. □

For any $a \in L$, let $a^{\Delta} = \bigwedge\{0_m \mid m$ is a maximal element containing $a\}$.

Lemma 1. *Let L be a reduced M-normal lattice. Then for any $a \in L$, a^{Δ} is a σ-element.*

P r o o f. Assume $x \in L_*$ and $x \leqslant a^\Delta$. Then $m \vee (0 : x) = 1$ for all maximal elements m containing a, so $(0 : x) \vee a = 1$. Therefore $y \vee a = 1$ for some compact element $y \leqslant (0 : x)$. Since $xy = 0$ and L is M-normal, by theorem 7 of [6] we have $(0 : x) \vee (0 : y) = 1$. Then $x_1 \vee y_1 = 1$ for some compact elements $x_1 \leqslant (0 : x)$ and $y_1 \leqslant (0 : y)$. Note that if m is a maximal element containing a, then $y \nleqslant m$ and so $y_1 \leqslant 0_m$. Therefore $y_1 \leqslant a^\Delta$ and obviously $a^\Delta \vee (0 : x) = 1$. This shows that a^Δ is a σ-element. □

Lemma 2. *Let L be a reduced M-normal lattice. Suppose a is a σ-element and let m be a maximal element containing a. If "$x \leqslant 0_m$ implies $x^\Delta \leqslant a$", then $a = 0_m$.*

P r o o f. Since $a \leqslant m$ and a is a σ-element, it follows that $a_m = 0_m$ and so $a \leqslant 0_m$. Assume $x \in L_*$ and $x \leqslant 0_m$. As 0_m is a $*$-element and therefore a σ-element, we have $0_m \vee (0 : x) = 1$, so $0_m \vee y = 1$ for some $y \in L_*$ with $xy = 0$. As L is M-normal, as in the proof of Lemma 1, we have $(0 : x) \vee (0 : y) = 1$, so $1 = x_1 \vee y_1$, where $xx_1 = yy_1 = 0$ for some $x_1, y_1 \in L_*$. Since $yy_1 = 0$ it follows that $y_1 \leqslant 0_m$. Therefore, by hypothesis $y_1^\Delta \leqslant a$. Again since $x \leqslant y_1^\Delta$, it follows that $x \leqslant a$ and hence $a = 0_m$. □

Theorem 4. *Let L be a reduced M-normal lattice.*
(i) *An element p is a minimal prime if and only if p is a maximal σ-element.*
(ii) *Every prime σ-element is a maximal σ-element.*

P r o o f. (i) Assume that p is a maximal σ-element. Suppose $p \leqslant m$ for some maximal element m of L. By Proposition 1, $p_m = 0_m$. As L is M-normal, 0_m is a minimal prime element and therefore (Theorem 1) a maximal σ-element. As $p \leqslant p_m$, it follows from the hypothesis on p that $p = p_m$, and hence that p is a minimal prime. The converse is given by Theorem 1.

(ii) Suppose a is a prime σ-element that is not a maximal σ-element. Then there is a maximal element m such that $a \leqslant m$ and $a \neq 0_m$. As a is a σ-element, $a \leqslant 0_m$. By Lemma 2, there exists $x \in L_*$ such that $x \leqslant 0_m$ and $x^\Delta \nleqslant a$. Note that $x^\Delta \wedge (0 : x)^\Delta = 0$. As a is a prime σ-element, it follows by Lemma 1 that $(0 : x)^\Delta \leqslant a$. Again since $x \leqslant 0_m$ and 0_m is a $*$-element and therefore a σ-element, we have $0_m \vee (0 : x) = 1$. So there exists $y \in L_*$ such that $y \leqslant 0_m$ and $y \nleqslant p$ for all maximal elements $p \geqslant (0 : x)$. As $y \leqslant 0_m$ and 0_m is a σ-element, it follows that $0_m \vee (0 : y) = 1$. So $z \vee y_1 = 1$ for some compact elements $z, y_1 \in L$ such that $z \leqslant 0_m$ and $yy_1 = 0$. Note that $y_1 \leqslant (0 : x)^\Delta$, so $m \vee (0 : x)^\Delta = 1$. But $(0 : x)^\Delta \leqslant a \leqslant m$, so $m = 1$, a contradiction. Thus a is a maximal σ-element. □

Corollary 1. *L is regular if and only if L is reduced and every prime element is a prime σ-element.*

P r o o f. If L is regular, then by Theorem 3, every prime element is a prime σ-element. Assume $x \in L_*$ and x is nilpotent. Then for every prime p, $x \leqslant p$ and $p \vee (0 : x) = 1$. It follows that $x = 0$, so L is reduced.

Conversely, if L is reduced and every prime is a σ-element, then by Theorem 1, every prime is a maximal σ-element, and so by Theorem 4, every prime element is a maximal element. If $x \in L_*$, then by Proposition 2, $x \vee (0 : x) = 1$, so L is a regular lattice. □

Theorem 5. *Let L be reduced. Then L is a Baer lattice if and only if every prime Baer element is a prime σ-element.*

P r o o f. If L is a Baer lattice, then by Theorem 2, every prime Baer element is a prime σ-element. Conversely, assume that every prime Baer element is a prime σ-element. Observe that a prime element which is a σ-element is a minimal prime element and therefore, by hypothesis, every prime Baer element is a minimal prime element and every minimal prime element is σ-element. Consequently by Theorem 3 of [6], L is quasiregular. It is observed in [6](p. 63) that every minimal prime is a Baer element, so by Theorem 1, L is M-normal as well as quasiregular. Fix $x \in L_*$. Choose an element $y \in L_*$ satisfying $(0 : (0 : x)) = (0 : y)$. Then $xy = 0$. It follows by Theorem 7 of [6] that $(0 : x) \vee (0 : y) = 0$. Hence $(0 : x) \vee (0 : (0 : x)) = 1$. Therefore L is a Baer lattice. □

Definition 3. L is said to be an almost Baer lattice if, for each $x \in L_*$, $(0 : x)$ is the join of complemented elements of L.

If R is an almost PP-ring (for each $a \in R$, aR is a projective R module), then the lattice $L(R)$ of all ideals of R is an almost Baer lattice (see [2]). If L_0 is a complementedly normal lattice, then the lattice $I(L_0)$ of all ideals of L_0, is an almost Baer lattice (see [8]). Every Baer lattice is an almost Baer lattice and an almost Baer lattice is a Baer lattice if and only if for each $x \in L_*$, there is a smallest complemented element y such that $x = xy$.

We record the following without proof.

Lemma 3. *L is an almost Baer lattice if and only if for each $x \in L_*$ and for any $y \in L_*$, $x \leqslant (0 : y)$ implies $xg = x$ for some complemented element $g \leqslant (0 : y)$.*

Definition 4. An element $a \in L$ is said to be a strong σ-element if for each $x \in L_*$, $x \leqslant a$ implies $e \vee (0 : x) = 1$ for some complemented element $e \leqslant a$.

Note that every strong σ-element is a σ-element.

Theorem 6. *Let L be reduced. Then the following statements are equivalent:*

(i) *L is an almost Baer lattice.*

(ii) *Every $*$-element is a strong σ-element.*

(iii) *Every minimal prime element is a strong σ-element.*

P r o o f. (i) $\Rightarrow$ (ii). Suppose (i) holds. Let $a = 0_F$ for some $F \in \mathfrak{F}(L_*)$. Let $x \leqslant a$ be any compact element. Then $x \leqslant (0 : y)$ for some $y \in F$. By (i) and Lemma 3, $xe = x$ for some complemented element $e \leqslant (0 : y)$. Note that $e \leqslant a$ and $e \vee (0 : x) = 1$ and therefore (ii) holds.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Assume that each minimal prime element is a strong σ-element. Observe that by Theorem 1, L is M-normal, so for every maximal element m, 0_m is a minimal prime element. Assume $x, y \in L_*$ and $x \leqslant (0 : y)$. We show that, for any maximal element m of L, there exists a complemented element $e' \nleqslant m$ such that either $e' \leqslant (0 : x)$ or $e' \leqslant (0 : y)$. Let m be a maximal element. Since 0_m is a minimal prime element we have either $x \leqslant 0_m$ or $y \leqslant 0_m$. As 0_m is a strong σ-element, there exists a complemented element $e \leqslant 0_m$ such that $xe = x$ or $ye = y$. Note that $(0 : e) = e' \nleqslant m$ and either $e' \leqslant (0 : x)$ or $e' \leqslant (0 : y)$. It follows that $1 = \bigvee\{f_\alpha \mid f_\alpha$ is a complemented element such that $f_\alpha \leqslant (0 : x)$ or $f_\alpha \leqslant (0 : y)\}$. As 1 is compact, $1 = \bigvee\limits_{i=1}^{n} f_{\alpha_i}$. Let $f_{\alpha_1}, f_{\alpha_2}, \ldots, f_{\alpha_k} \leqslant (0 : x)$ and $f_{\alpha_{k+1}}, f_{\alpha_{k+2}}, \ldots, f_n \leqslant (0 : y)$. Put $g = \bigvee\limits_{i=k+1}^{n} f_{\alpha_i}$. Then $xg = x$ and $g \leqslant (0 : y)$. This shows that L is an almost Baer lattice and the proof is complete. □

Let $c(L) = \{x \in L \mid x$ is a complemented element$\}$ and let $R(L) = \{a \in L \mid a$ is the join of complemented elements of $L\}$. Then $R(L) = \big(R(L), \bigwedge\limits_R, \bigvee, 0, 1\big)$ is a regular lattice, where for any collection $\{a_\alpha\} \subseteq R(L)$, $\bigwedge\limits_R a_\alpha = \bigvee\{x \in c(L) \mid x \leqslant a_\alpha$ for all $\alpha\}$. Note that for $a_1, a_2, \ldots, a_n \in R(L)$, $\bigwedge\limits_{i=1}^{n} a_i = \bigwedge\limits_{\substack{R\\ i=1}}^{n} a_i = a_1 a_2 \ldots a_n$.

For any prime p of L we define $p_R = \bigvee\{a \in R(L) \mid a \leqslant p\}$. For any prime q of $R(L)$ we define $q^* = \bigvee\{x \in L_* \mid xe = 0$ for some complemented element $e \nleqslant q\}$. Note that $p_R \leqslant p$ and $q \leqslant q^*$.

Lemma 4. *Let p be a prime element of L. Then p_R is a prime element of $R(L)$.*

P r o o f. Obvious. □

Henceforth, we denote the complement of an element $x \in c(L)$ by x'.

Lemma 5. *Let L be a reduced almost Baer lattice and let q be a prime element of $R(L)$. Then q^* is minimal prime of L and a prime σ-element.*

P r o o f. Suppose $x, y \in L_*$ and let $xy \leqslant q^*$. Then $xye = 0$ for some complemented element $e \nleqslant q$. Assume that $y \nleqslant q^*$. As L is an almost Baer lattice, we have $xf = x$ and $f \leqslant (0 : ye)$ for some $f \in C(L)$. Since $yfe = 0$, $y \nleqslant q^*$, it follows that $f \leqslant q$, so $f' \nleqslant q$ and also $xf' = 0$. Therefore $x \leqslant q^*$. This shows that q^* is a prime element and since $q^* = 0_{q^*}$, it follows that q^* is a minimal prime element. As L is M-normal, by Theorem 1, q^* is a minimal prime in L and a prime σ-element. □

Let $\pi(R(L))$ be the set of prime elements of $R(L)$ and $\pi(\sigma(L))$ be the set of prime σ-elements of L.

Theorem 7. *Let L be a reduced almost Baer lattice. Then the map $q \longrightarrow q^*$ from $\pi(R(L))$ into $\pi(\sigma(L))$ is a bijection map.*

P r o o f. Suppose $q^* = p^*$ for some p, $q \in \pi(R(L))$. We show that $q \leqslant p$. Assume $x \in L_* \cap c(L)$ and $x \leqslant q$. Then there exists a complemented element e with $x \leqslant e \leqslant q$. Necessarily $e' \nleqslant q$, so $e \leqslant q^* = p^*$. Hence also $e' \nleqslant p^*$. As $e \nleqslant p$ implies $e' \leqslant p^*$it follows that $e \leqslant p$. Hence $x \leqslant e \leqslant p$. Hence $q \leqslant p$. Similarly $p \leqslant q$ and hence $p = q$. Therefore the map is one-one. If $p \in \pi(\sigma(L))$, then by Lemma 5, p is a minimal prime element, so by Lemma 4, $p_R \in \pi(R(L))$. Again by Lemma 5, $p_R^* \in \pi(\sigma(L))$ and $p_R^* \leqslant p$ and hence $p_R^* = p$. Thus the map is a bijection. □

Definition 5. L is said to be relatively M-normal if any two noncomparable prime elements are comaximal. (a, $b \in L$ are said to be comaximal if $a \vee b = 1$).

Note that regular lattices, zero dimensional lattices are examples of relatively M-normal lattices. If R is a Prüfer domain, then the lattice $L(R)$ of all ideals of R is a relatively M-normal lattice (see [10]). If L_0 is a relatively normal lattice (see [8]), then $I(L_0)$ is a relatively M-normal lattice. If L is an r-lattice domain satisfying any one of the conditions of Theorem 3.4 of [4], then L is a relatively M-normal lattice.

We record the following four lemmas for future reference.

Lemma 6. *The following statements are equivalent for an element $a \in L$.*

(i) $(a : x) = (a : x^n)$ *for all $x \in L_*$ and for all $n \in \mathbb{Z}^+$.*

(ii) $a = \sqrt{a}$.

(iii) $(a : xy) = (a : x \wedge y)$ *for all x, $y \in L_*$.*

P r o o f. The implications $(i) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ are easily established. □

Lemma 7. *Let $a \in L$. If $a = \sqrt{a}$, then for any $x \in L_*$, $(a : x) = a_{[x)}$.*

P r o o f. Clearly $x \in [x)$, so $(a : x) \leqslant a_{[x)}$. If $t \leqslant a_{[x)}$, then $ty \leqslant a$ for some $y \geqslant$ some power of x. It follows that $t^n x^n \leqslant tx^n \leqslant a$ for some n, and hence that $t \leqslant (a : x)$. □

Lemma 8. *Let p be a minimal prime over a. Then, for any $x \in L_*$, p contains precisely one of x, $(\sqrt{a} : x)$.*

Proof. Suppose $x \in L_*$. As $(\sqrt{a} : x)x \leqslant \sqrt{a} \leqslant p$, it follows that either $x \leqslant p$ or $(\sqrt{a} : x) \leqslant p$. If $x \leqslant p$, then by Proposition 2 there exists a compact element $y \nleqslant p$ such that $x^n y \leqslant a$ for some $n \in \mathbb{Z}^+$. As $(xy)^n \leqslant a$, we have $y \leqslant (\sqrt{a} : x)$ and therefore $(\sqrt{a} : x) \nleqslant p$. This shows that p contains precisely one of x, $(\sqrt{a} : x)$. $\square$

Lemma 9. *Assume $a \in L$, $F \in \mathfrak{F}(L_*)$ and $a_f \neq 1$. If p is a minimal prime over a_F then p is a minimal prime over a.*

Proof. Suppose p is a minimal prime over a_F. Obviously $a \leqslant p$. Let x be any compact element such that $x \leqslant p$. By Proposition 2, we get the following: As p is a minimal prime over a_F, there exists a compact $y \nleqslant p$, such that $x^n y \leqslant a_F$ for some $n \in \mathbb{Z}^+$. Then $x^n ys \leqslant a$ for some $s \in F$. Note that $[0, b] \cap F = [0, p] \cap F = \emptyset$, so $s \nleqslant p$. Thus $ys \nleqslant p$ and $x^n ys \leqslant a$, and hence p is a minimal prime over a. $\square$

Theorem 8. *Let a be a proper element of L. Then the following statements are equivalent:*

(i) *For any x, $y \in L_*$, $xy \leqslant a$ implies $(a : x) \vee (a : y) = 1$.*

(ii) *For every x, $y \in L_*$, $(a : xy) = (a : x) \vee (a : y)$.*

(iii) *$a = \sqrt{a}$ and for every prime element p containing a, a_p is a prime element.*

(iv) *$a = \sqrt{a}$ and every prime element containing a, contains a unique minimal prime over a.*

(v) *$a = \sqrt{a}$ and any two distinct minimal primes over a are comaximal.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Let x, $y \in L_*$. Clearly $(a : x) \vee (a : y) \leqslant (a : xy)$. Choose any compact element $r \in L_*$ such that $r \leqslant (a : xy)$. Then $xyr \leqslant a$, so by (i), $(a : x) \vee (a : yr) = 1$. Again $r = 1r = ((a : x) \vee (a : yr))r = (a : x)r \vee (a : yr)r = (a : x)r \vee (((a : y) : r))r \leqslant (a : x) \vee (a : y)$ as $(a : x)r \leqslant (a : x)$ and $((a : y) : r)r \leqslant (a : y)$. Thus $(a : x) \vee (a : y) = (a : xy)$.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. By (ii), $(a : x^n) = (a : x)$ for every $x \in L_*$ and for every positive integer n and so by Lemma 6, $a = \sqrt{a}$. Let p be a prime element containing a. Suppose $xy \leqslant a_p$ for some x, $y \in L_*$. Then $xys \leqslant a$ for some $s \nleqslant p$. Suppose $x \nleqslant a_p$. Then $xz \nleqslant a$ for all $z \in F_p$, so $(a : xs) \leqslant p$. By (ii), $1 = (a : y) \vee (a : xs)$ and hence $(a : y) \nleqslant p$. Therefore there exists $r \in L_*$ such that $yr \leqslant a$ and $r \nleqslant p$. As $r \nleqslant p$, necessarily $y \leqslant a_p$. This shows that a_p is a prime element.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds. Note that $a \leqslant a_p$ for every prime element p of L. Again if p, q are prime elements such that $a \leqslant q \leqslant p$, then $a_p \leqslant a_q \leqslant q$. Therefore if

p is a prime element containing a, then by (iii), a_p is the only minimal prime over a that is contained in p. Thus (iv) holds.

(iv) $\Leftrightarrow$ (v) is obvious.

(iv) $\Rightarrow$ (i). Suppose (iv) holds. Assume $x, y \in L_*$ and $xy \leqslant a$. If a is a radical element, by Lemma 7, $(a : x) = a_{[x)}$ and $(a : y) = a_{[y)}$. Suppose $(a : x) \vee (a : y) < 1$. Then $(a : x) \vee (a : y) \leqslant p$ for some prime element p of L. Again there exist prime elements $p_1, p_2 \in L$ such that $(a : x) \leqslant p_1 \leqslant p$, $(a : y) \leqslant p_2 \leqslant p$, p_1 is a minimal prime over $(a : x)$ and p_2 is a minimal prime over $(a : y)$. By Lemma 9, p_1 and p_2 are minimal primes over a and so by (iv), $p_1 = p_2$. By Lemma 8, $x \nleqslant p_1$ and $y \nleqslant p_1$ and hence $xy \nleqslant p_1$, which contradicts the fact the $xy \leqslant a \leqslant p_1$. Therefore $(a : x) \vee (a : y) = 1$ and hence (i) holds. This completes the proof of the theorem. $\square$

We now characterize relatively M-normal lattices.

Theorem 9. *The following statements on L are equivalent:*

(i) *For every $x, y, a \in L_*$, $xy \leqslant \sqrt{a}$ implies $(\sqrt{a} : x) \vee (\sqrt{a} : y) = 1$.*

(ii) *For every $x, y, a \in L_*$, $(\sqrt{a} : xy) = (\sqrt{a} : x) \vee (\sqrt{a} : y)$.*

(iii) *For every prime element p and $a \leqslant p$, $(\sqrt{a})_p$ is a prime element.*

(iv) *Every prime element containing an element $a \in L$ contains a unique minimal prime over a.*

(v) *Any two distinct minimal primes over an element $a \in L$ are comaximal.*

(vi) *For every $x, y \in L_*$, $(\sqrt{x} : y) \vee (\sqrt{y} : x) = 1$.*

(vii) *L is a relatively M-normal lattice.*

Proof. By Theorem 8, (i) through (v) are equivalent. We show that (i), (vi) and (vii) are equivalent.

(i) $\Rightarrow$ (vi). Suppose (i) holds. Let $x, y \in L_*$. Then $1 = (\sqrt{x} \wedge \sqrt{y} : x \wedge y) = (\sqrt{xy} : x \wedge y) =$(Lemma6) $(\sqrt{xy} : xy) = (\sqrt{xy} : x) \vee (\sqrt{xy} : y)$. But $(\sqrt{xy} : x) = (\sqrt{y} : x)$ and $(\sqrt{xy} : y) = (\sqrt{x} : y)$ and therefore $1 = (\sqrt{x} : y) \vee (\sqrt{y} : x)$. Thus (vi) holds.

(vi) $\Rightarrow$ (vii). Suppose (vi) holds. Let p_1, p_2 by any two incomparable prime elements. Choose $x, y \in L_*$ such that $x \leqslant p_1$, $x \nleqslant p_2$, $y \leqslant p_2$ and $y \nleqslant p_1$. Then $(\sqrt{x} : y) \leqslant p_1$ and $(\sqrt{y} : x) \leqslant p_2$. Therefore by (vi), p_1 and p_2 are comaximal. Hence (vii) holds.

The proof of (vii) $\Rightarrow$ (i) is similar to the proof of Theorem 8 ((iv) $\Rightarrow$ (i)).

Thus (i), (vi) and (vii) are equivalent. $\square$

Remark 2. By definition, every relatively M-normal lattice is an M-normal lattice. By Theorem 9(vi), L is a relatively M-normal lattice if and only if any two radical elements are locally comparable. This shows that if every compact element is principal, then L is a relatively M-normal lattice and L_m is totally ordered for every maximal element m of L (see Theorem 4 and Theorem 6 of [11]). We are unable to prove the converse. It would be interesting to find some conditions for a relatively M-normal lattice to be a lattice in which every compact element is principal.

References

[1] *H. Al-Ezeh*: The pure spectrum of a PF-ring, Commen. Math. Univer. Sancti Pauli.
[2] *H. Al-Ezeh*: Further results on reticulated rings. Act. Math. Hung. *60* (1992), 1–6.
[3] *F. Alarcon, D.D. Anderson and C. Jayaram*: Some results on commutative ideal theory. Period. Math. Hung. *30* (1995), 1–26.
[4] *D.D. Anderson*: Abstract commutative ideal theory without chain condition. Algebra Universalis *6* (1976), 131–145.
[5] *D.D. Anderson and C. Jayaram*: Regular lattices. Studia Sci. Math. Hung. *30* (1995), 379–388.
[6] *D.D. Anderson, C. Jayaram and P.A. Phiri*: Baer lattices. Act. Sci. Math. (Szeged) *59* (1994), 61–74.
[7] *F. Borceux and G. Van de Bossche*: Algebra in a localic topos with applications to ring theory. Lecture Notes in Mathematics No. 1038, Spring Verlag, Berlin–Heidelberg, 1983.
[8] *W.H. Cornish*: Normal lattices. J. Aust. Math. Soc. *14* (1972), 200–215.
[9] *W.H. Cornish*: 0-ideals, congruences and sheaf representations of distributive lattices. Rev. Roumaine. Math. Pure Appl. *22* (1977), 1059–1067.
[10] *C.U. Jensen*: On characterizations of Prüfer rings. Math. Scand. *13* (1963), 90–98.
[11] *P.J. McCarthy*: Arithmetical rings and multiplicative lattices. Ann. Mat. Pura. Appl. *82* (1969), 267–276 MR 40 # 1378.

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