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ANTIATOMIC RETRACT VARIETIES OF MONOUNARY ALGEBRAS

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Retracts of monounary algebras were investigated in the papers [2]–[4].

The notion of the retract variety of monounary algebras was introduced in [5] by applying an analogy with the notion of the order variety of partially ordered sets studied in [1].

The collection $\mathfrak{R}$ of all retract varieties of monounary algebras was investigated in [5]. This collection is considered to be partially ordered by the class-theoretical inclusion. A retract variety $\mathscr{V}$ is called atomic if $\mathscr{V} \neq \emptyset$ and, whenever $\mathscr{V}'$ is a retract variety with $\emptyset \neq \mathscr{V}' \subseteq \mathscr{V}$, then $\mathscr{V}' = \mathscr{V}$. It was proved that there are exactly $2^{\aleph_0}$ atomic retract varieties in $\mathfrak{R}$.

A retract variety $\mathscr{V}$ of $\mathfrak{R}$ is said to be antiatomic if $\mathscr{V} \neq \emptyset$ and there is no atomic variety $\mathscr{V}_1$ of $\mathfrak{R}$ with $\mathscr{V}_1 \subseteq \mathscr{V}$.

In view of the relation $\subseteq$ for pairs of retract varieties $\mathscr{V}_1, \mathscr{V}_2$ of $\mathfrak{R}$, we apply also the symbols inf $\{\mathscr{V}_1, \mathscr{V}_2\}$ and sup $\{\mathscr{V}_1, \mathscr{V}_2\}$ in the usual way. Namely, if $\mathscr{V}_1$, $\mathscr{V}_2$, $\mathscr{V}_3$ belong to $\mathfrak{R}$ and

(i) $\mathscr{V}_1 \subseteq \mathscr{V}_3$, $\mathscr{V}_2 \subseteq \mathscr{V}_3$,

(ii) if $\mathscr{V}$ belongs to $\mathfrak{R}$, $\mathscr{V}_1 \subseteq \mathscr{V}$, $\mathscr{V}_2 \subseteq \mathscr{V}$, then $\mathscr{V}_3 \subseteq \mathscr{V}$,

then we write $\mathscr{V}_3 = \sup\{\mathscr{V}_1, \mathscr{V}_2\}$. The notion inf $\{\mathscr{V}_1, \mathscr{V}_2\}$ is defined dually.

The description of all antiatomic retract varieties of $\mathfrak{R}$ is given in Theorem 3.2. Further we investigate the collection *Ant* of all antiatomic retract varieties of $\mathfrak{R}$; the following results will be proved:

(a) *Ant* is closed with respect to the operations of inf and sup.

(b) There is a proper class $\mathscr{O}$ of ordinals with the following properties:

(b1) For each $\alpha \in \mathscr{O}$ there exists $\mathscr{W}_\alpha \in Ant$ such that, whenever $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$, then $\mathscr{W}_\alpha \nsubseteq \mathscr{W}_\beta$.

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(b2) For each $\alpha \in \mathscr{O}$ there exists $\mathscr{V}_\alpha \in Ant$ such that, whenever $\alpha, \beta \in \mathscr{O}$, $\beta < \alpha$, then $\mathscr{V}_\beta \subsetneqq \mathscr{V}_\alpha$.

1. Preliminaries

The symbol $\mathscr{U}$ will denote the class of all monounary algebras. Let $\mathscr{A} = (A, f) \in \mathscr{U}$. A nonempty subset M of A is said to be a retract of $\mathscr{A}$ if there is a mapping h of A onto M such that h is an endomorphism of $\mathscr{A}$ and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M.

For $\mathscr{K} \subseteq \mathscr{U}$ let $R(\mathscr{K})$ $(P(\mathscr{K}))$ be the class of monounary algebras whose elements are all retracts (direct products) of members of $\mathscr{K}$ and their isomorphic images. A class $\mathscr{K}$ of monounary algebras is said to be retract (product) closed if it is closed with respect to isomorphisms and if it contains all retracts (direct products) of members of $\mathscr{K}$. A class $\mathscr{K}$ is said to be a retract variety if it is retract closed and product closed. By a retract variety $\mathscr{V}(\mathscr{K})$ generated by $\mathscr{K}$ (cf. [5]) we understand the class of all monounary algebras such that any of them is a member of every retract variety $\mathscr{C}$ such that $\mathscr{C} \supseteq \mathscr{K}$.

1.1. Lemma. ([5], 1.3) *If $\mathscr{K} \subseteq \mathscr{U}$, then $V(\mathscr{K}) = RP(\mathscr{K})$.*

In what follows we will use the notion of the degree of an element $x \in A$, where $(A, f) \in \mathscr{U}$; for this notion cf. e.g. [7], [6] and [2]. The degree of $x \in A$ is an ordinal or the symbol ∞ and is denoted by $s_f(x)$.

We will use without quotation the following properties of $s_f(x)$:

(A) Let $(A, f) = \prod_{i \in I} (A_i, f), x \in A$. Then $s_f(x) \leqslant s_f(x(i))$ for each $i \in I$.

(B) If φ is a homomorphism of (A, f) into (B, g) and $x \in A$, then $s_g(\varphi(x)) \geqslant s_f(x)$.

1.2. Definition. Let (A, f) be a connected monounary algebra. We say that (A, f) is unbounded, if

(i) $s_f(x) \neq \infty$ for each $x \in A$,

(ii) if $x \in A$, $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^{-(m+n)}(f^m(x)) \neq \emptyset$.

1.3. Lemma. *Let (A, f) be a connected monounary algebra. Then (A, f) is unbounded if and only if*

(i) *$s_f(x) \neq \infty$ for each $x \in A$,*

(ii′) *there is $x \in A$ such that if $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ with $f^{-(m+n)}(f^m(x)) \neq \emptyset$.*

P r o o f. Obviously, if (i) and (ii) hold, then (i) and (ii′) hold. Suppose that (i) and (ii′) are valid. Let $x_1 \in A$, $n_1 \in \mathbb{N}$. The algebra (A, f) is connected, thus there are $k, l \in \mathbb{N} \cup \{0\}$ with

$$f^k(x) = f^l(x_1). \tag{1}$$

There is $j \in \mathbb{N}$ such that $l - k + j \geqslant 0$. Put $n = n_1 + l - k + j$. According to (ii′), there are $m \in \mathbb{N}$ and $y \in A$ such that

$$f^{m+n}(y) = f^m(x). \tag{2}$$

Denote $m_1 = m + l - k + j$. Then $m_1 \in \mathbb{N}$ and (1) and (2) imply

$$\begin{aligned} f^{m_1}(x_1) &= f^{m+l-k+j}(x_1) = f^{m+j}(x) \\ &= f^{m+n+j}(y) = f^{(m_1-l+k-j)+(n_1+l-k+j)+j}(y) \\ &= f^{m_1+n_1}(f^j(y)). \end{aligned}$$

Therefore

$$f^{-(m_1+n_1)}(f^{m_1}(x_1)) \neq \emptyset, \tag{3}$$

which implies that (ii) is valid. □

Remark. It can be shown that the conditions (ii) and (ii′) are equivalent for each connected monounary algebra.

We will apply the following notation introduced in [5]:

1.4. Notation. *Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Z}$ the set of all integers.*
$\underline{\mathbb{Z}} = (\mathbb{Z}, f)$, where $f(k) = k + 1$ for each $k \in \mathbb{Z}$,
$\underline{\mathbb{N}} = (\mathbb{N}, f)$, where $f(k) = k + 1$ for each $k \in \mathbb{N}$.

1.5. Lemma. *Let (A, f) be a monounary algebra such that $s_f(x) \neq \infty$ for each $x \in A$. If (B, f) is a connected component of (A, f), then (B, f) fails to be unbounded if and only if there exist distinct elements $e_k \in B$ for $k \in \mathbb{N}$ such that $f(e_k) = e_{k+1}$ and $f^{-k}(e_k) = \emptyset$ for each $k \in \mathbb{N}$.*

P r o o f. Suppose that (B, f) is a connected component of (A, f) and that (B, f) fails to be unbounded. Then in view of 1.2, either

$$s_f(x) = \infty \quad \text{for some } x \in B \tag{1}$$

or

(2) there are $x \in B$, $n \in \mathbb{N}$ such that if $m \in \mathbb{N}$, then

$$f^{-(m+n)}(f^m(x)) = \emptyset.$$

The assumption yields that (2) is valid. We can suppose that n is the least positive integer with the above property, i.e., that

(2′) there are $x \in B, n \in \mathbb{N}$ such that if $m \in \mathbb{N}$, then

$$f^{-(m+n)}(f^m(x)) = \emptyset \text{ and } f^{-(m+n-1)}(f^m(x)) \neq \emptyset.$$

Let x and n satisfy (2′). Take a fixed $m \in \mathbb{N}$ and an arbitrary element

$$e_1 \in f^{-(m+n-1)}(f^m(x)).$$

Further, put

(3) $$e_k = f^{k-1}(e_1) \text{ for each } k \in \mathbb{N}, k > 1.$$

The elements e_k (for $k \in \mathbb{N}$) are then distinct by (1) and

(4) $$f(e_k) = e_{k+1} \text{ for each } k \in \mathbb{N}.$$

Suppose that there is $y \in f^{-k}(e_k)$ for some $k \in \mathbb{N}$. We get

$$f^k(y) = e_k = f^{k-1}(e_1) \in f^{-n}(f^k(x)),$$
$$y \in f^{-(k+n)}(f^k(x)),$$

a contradiction to (2). Hence $f^{-k}(e_k) = \emptyset$ for each $k \in \mathbb{N}$.

Conversely, let there exist distinct elements $\{e_k \colon k \in \mathbb{N}\}$ with $f(e_k) = e_{k+1}$ and $f^{-k}(e_k) = \emptyset$ for each $k \in \mathbb{N}$. If we take $x = e_1, n = 1$, then, for $m \in \mathbb{N}$,

$$\begin{aligned} f^{-(m+1)}(f^m(x)) &= f^{-(m+1)}(f^m(e_1)) \\ &= f^{-(m+1)}(e_{m+1}) = \emptyset. \end{aligned}$$

Therefore (B, f) satisfies (2) and we obtain that (B, f) fails to be unbounded. □

1.6. Corollary. *Let (A, f) be a monounary algebra such that $s_f(x) \neq \infty$ for each $x \in A$. If (B, f) is a connected component of (A, f), then (B, f) is unbounded if and only if $\underline{\mathbb{N}} \notin R(B, f)$.*

P r o o f. The assertion is a consequence of 1.5 and of [3], 3.2. □

1.7. Lemma. *Let (A,f) be a monounary algebra such that $s_f(x) \neq \infty$ for each $x \in A$. The following conditions are equivalent:*

(i) *if (B,f) is a connected component of (A,f), then (B,f) fails to be unbounded;*

(ii) $\underline{\mathbb{N}} \in R(A,f)$.

Proof. Suppose that $\underline{\mathbb{N}} \in R(A,f)$. Then there is a subalgebra (E,f) of (A,f) with $(E,f) \cong \underline{\mathbb{N}}$ such that E is a retract of (A,f).

Hence $E = \{e_n \colon n \in \mathbb{N}\}$, where $e_i \neq e_j$ for $i,j \in \mathbb{N}, i \neq j$ and $f(e_n) = e_{n+1}$ for each $n \in \mathbb{N}$. Let (B,f) be a connected component of (A,f). Then there is an endomorphism φ of (B,f) into (E,f) and there is $i \in \mathbb{N}$ such that

$$\varphi^{-1}(e_j) \cap B = \emptyset \text{ for each } j \in \mathbb{N}, j < i, \tag{1}$$

$$\varphi^{-1}(e_j) \cap B \neq \emptyset \text{ for each } j \in \mathbb{N}, j \geqslant i. \tag{2}$$

Denote by b_1 an arbitrary element of $\varphi^{-1}(e_i) \cap B$ and, for each $n \in \mathbb{N}, n > 1$, put $b_k = f^{k-1}(b_1)$. Obviously,

$$f(b_k) = b_{k+1} \text{ for each } k \in \mathbb{N}. \tag{3}$$

Let $k \in \mathbb{N}$ and suppose that there is $y \in f^{-k}(b_k)$. Then $f^k(y) = b_k$, which implies

$$\begin{aligned} f^k(\varphi(y)) = \varphi(f^k(y)) = \varphi(b_k) = \varphi(f^{k-1}(b_1)) \\ = f^{k-1}(\varphi(b_1)) = f^{k-1}(e_i) = f^{k-1}(f^{i-1}(e_1)) \\ = f^{k+i-2}(e_1). \end{aligned}$$

Thus

$$\varphi(y) \in f^{i-2}(e_1),$$

which yields that $i > 1$ and

$$\varphi(y) = e_{i-1}.$$

Therefore

$$y \in \varphi^{-1}(e_{i-1}) \cap B,$$

which contradicts (1). According to (3) and 1.5 we obtain that (i) is valid.

Now let (i) hold. By 1.6, for each connected component (B,f) of (A,f) we have $\underline{\mathbb{N}} \in R(B,f)$, hence $\underline{\mathbb{N}} \in R(A,f)$. $\square$

In 2.1 – 2.6 we suppose that $\mathscr{V} \in \mathfrak{R}$ and that there exist $\mathscr{A} = (A, f) \in \mathscr{V}$ and $x_0 \in A$ such that $s_f(x_0) = \infty$.

The definition of $s_f(x)$ implies

2.1. Lemma. *There is a subalgebra of $\mathscr{A}$ such that either it is isomorphic to $\underline{\mathbb{Z}}$ or it is a cycle.*

2.2. Lemma. *Assume that each connected component of $\mathscr{A}$ is a cycle. Further suppose that*

(i) *if $(B, f), (C, f)$ are cycles of $\mathscr{A}$, card $B \neq$ card C, then card B does not divide card C.*

Then $\mathscr{V} \notin Ant$.

P r o o f. Let the assumption hold. There is a subalgebra (E, f) of (A, f) such that

(1) $$\text{if } (B, f), (C, f) \text{ are cycles of } (E, f) \text{ then card } B \neq \text{ card } C;$$

(2) $$\text{if } (B, f) \text{ is a cycle of } \mathscr{A}, \text{ then there is a cycle } (C, f) \text{ of } (E, f) \text{ with card } B = \text{ card } C.$$

By [2], 1.3 we obtain

(3) $$(E, f) \in R(A, f),$$

which implies

(4) $$V(E, f) \subseteq V(A, f) \subseteq \mathscr{V}.$$

Further, (E, f) satisfies (i), hence [5], 3.9 yields that $V(E, f)$ is atomic. According to (4) we get that $\mathscr{V} \notin Ant$. □

2.3. Lemma. *If each connected component of $\mathscr{A}$ is a cycle, then $\mathscr{V} \notin Ant$.*

P r o o f. Suppose that each connected component of $\mathscr{A}$ is a cycle. For $x \in A$ let $n(x)$ be the cardinality of the cycle $\mathscr{C}(x)$ with $x \in \mathscr{C}(x)$. Further let E be the set of all $x \in A$ such that if $y \in A - \mathscr{C}(x)$, then either $n(y) = n(x)$ or $n(y)$ does not divide $n(x)$. The algebra (E, f) satisfies the condition (i) of 2.2, hence

$$V(E, f) \notin Ant. \tag{1}$$

Further, the definition of (E, f) implies (according to [2],1.3) that we have

$$(E, f) \in R(A, f). \tag{2}$$

By (2), $V(E, f) \subseteq V(A, f) \subseteq \mathscr{V}$, hence (1) yields that $\mathscr{V} \notin Ant$. $\square$

2.4. Lemma. *If each connected component of $\mathscr{A}$ contains a cycle and there is $x \in A$ which does not belong to any cycle, then $\mathscr{V} \notin Ant$.*

Proof. Let the assumption hold and let D be the set–theoretical union of all cycles of $\mathscr{A}$. Then [2], 1.3 implies $(D, f) \in R(A, f)$, thus

$$V(D, f) \subseteq V(A, f) \subseteq \mathscr{V}.$$

Further, $V(D, f) \notin Ant$ with respect to 2.3, hence we obtain that $\mathscr{V} \notin Ant$, either. $\square$

2.5. Lemma. *Suppose that (A, f) contains a subalgebra which is a cycle. Then $\mathscr{V} \notin Ant$.*

Proof. Let B be the set of all elements of connected components of (A, f) which contain a cycle. By the assumption, $B \neq \emptyset$. From 2.3 and 2.4 we get

$$V(B, f) \notin Ant. \tag{1}$$

Further, [2], 1.3 implies $(B, f) \in R(A, f)$, hence

$$V(B, f) \subseteq V(A, f) \subseteq \mathscr{V}. \tag{2}$$

Then (1) and (2) yield that $\mathscr{V} \notin Ant$. $\square$

2.6. Lemma. *Suppose that no subalgebra of (A, f) is a cycle. Then $\mathscr{V} \notin Ant$.*

Proof. By 2.1, there is a subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{B} \cong \underline{\mathbb{Z}}$. Then [2], 1.3 implies $\underline{\mathbb{Z}} \in R(\mathscr{A})$, thus

$$V(\underline{\mathbb{Z}}) \subseteq V(\mathscr{A}) \subseteq \mathscr{V}.$$

Since $V(\underline{\mathbb{Z}})$ is atomic, we obtain that $\mathscr{V} \notin Ant$. $\square$

2.7. Proposition. *If $\mathscr{V} \in \mathfrak{R}$ and there are $(A, f) \in \mathscr{V}$ and $x \in A$ with $s_f(x) = \infty$, then $\mathscr{V} \notin Ant$.*

Proof. Let the assumption hold. By 2.1, there is a subalgebra $\mathscr{B}$ of (A, f) such that either

$$\mathscr{B} \text{ is a cycle of } (A, f) \tag{1}$$

or

$$B \cong \underline{\mathbb{Z}}. \tag{2}$$

If (1) is valid, then 2.5 yields that $\mathscr{V} \notin Ant$. If no subalgebra of (A, f) is a cycle, then the required assertion is obtained by virtue 2.6. □

3. The collection *Ant*

In this section we will describe all antiatomic retract varieties of $\mathfrak{R}$. Further, it will be proved that Ant is closed with respect to the operations of inf and sup. It is obvious that if $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, then

$$\inf\{\mathscr{V}_1, \mathscr{V}_2\} = \mathscr{V}_1 \cap \mathscr{V}_2$$

and that inf $\{\mathscr{V}_1, \mathscr{V}_2\}$ belongs to Ant. We have to show that if $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, then sup $\{\mathscr{V}_1, \mathscr{V}_2\}$ belongs to Ant, as well.

3.1. Lemma. *Let $\mathscr{V}_1, \mathscr{V}_2 \in Ant$. Then sup $\{\mathscr{V}_1, \mathscr{V}_2\} = RP(\mathscr{V}_1 \cup \mathscr{V}_2)$.*

Proof. The assertion is a consequence of 1.1. □

3.2. Theorem. *Let $\mathscr{V} \in \mathfrak{R}$. The following conditions are equivalent:*

(i) *$\mathscr{V} \in Ant$;*
(ii) *if $(A, f) \in \mathscr{V}$, then $s_f(x) \neq \infty$ for each $x \in A$ and there is a connected component $\mathscr{B}$ of (A, f) such that $\mathscr{B}$ is unbounded.*

Proof. Let (i) hold and let (ii) be not valid. There exists $(A, f) \in \mathscr{V}$ such that either

$$\text{there is } x \in A \text{ with } s_f(x) = \infty \tag{1}$$

or

(2) if $\mathscr{B}$ is a connected component of (A, f), then $\mathscr{B}$ fails to be unbounded.

From 2.7 it follows that (1) is not valid, thus (2) holds. Then 1.7 implies that $\underline{\mathbb{N}} \in R(A, f)$, therefore

$$V(\underline{\mathbb{N}}) \subseteq V(A, f) \subseteq \mathscr{V}.$$

Since $V(\underline{\mathbb{N}})$ is atomic, we obtain that $\mathscr{V} \notin Ant$, a contradiction.

Conversely, suppose that (ii) is valid and that (i) fails to hold. Then there is an atomic retract variety $V(\mathscr{A})$ in $\mathfrak{R}$ such that $V(\mathscr{A}) \subseteq \mathscr{V}$. By (ii),

(1) $$s_f(x) \neq \infty \text{ for each } x \in A.$$

Therefore [5], 3.9 implies

(2) $$V(\mathscr{A}) = V(\underline{\mathbb{N}}).$$

According to (2) we have $\underline{\mathbb{N}} \in \mathscr{V}$. Moreover, $\underline{\mathbb{N}}$ is the unique connected component of $\underline{\mathbb{N}}$ and $\underline{\mathbb{N}}$ fails to be unbounded, which is a contradiction to (ii). □

3.3. Lemma. *Let $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, $(A, f) \in P(\mathscr{V}_1 \cup \mathscr{V}_2)$. Then $s_f(x) \neq \infty$ for each $x \in A$.*

P r o o f. By assumption, there are $(A_1, f) \in \mathscr{V}_1$ and $(A_2, f) \in \mathscr{V}_2$ with $(A, f) = (A_1, f) \times (A_2, f)$. Let $x = (x_1, x_2) \in A_1 \times A_2$. Then

(1) $$s_f(x) \leqslant s_f(x_1).$$

Since $x_1 \in A_1, (A_1, f_1) \in \mathscr{V}_1 \in Ant$, we obtain according to 3.2

(2) $$s_f(x_1) \neq \infty.$$

Hence (1) and (2) yield $s_f(x) \neq \infty$. □

3.4. Lemma. *Let the assumption of* 3.3 *hold. There is $x \in A$ such that if $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ with $f^{-(m+n)}(f^m(x)) \neq \emptyset$.*

P r o o f. Analogously as in 3.3, $(A, f) = (A_1, f) \times (A_2, f)$, where $(A_1, f) \in \mathscr{V}_1$, $(A_2, f) \in \mathscr{V}_2$. Further, 3.2 implies that there are connected components (B_1, f) of (A_1, f) and (B_2, f) of (A_2, f) which are unbounded. Hence

(1) there is $x_1 \in B_1$ such that if $n \in \mathbb{N}$, then there is $m_1 \in \mathbb{N}$ with $f^{-(m_1+n)}(f^{m_1}(x_1)) \neq \emptyset$.

Let the analogous assertion for B_2 be denoted by (2). Put

$$x = (x_1, x_2) \in B_1 \times B_2.$$

Let $n \in \mathbb{N}$. According to (1) and (2) there are $m_1, m_2 \in \mathbb{N}$ with

$$f^{-(m_1+n)}(f^m(x_1)) \neq \emptyset, f^{-(m_2+n)}(f^{m_2}(x_2)) \neq \emptyset;$$

take $y_1 \in f^{-(m_1+n)}(f^{m_1}(x_1))$, $y_2 \in f^{-(m_2+n)}(f^{m_2}(x_2))$. Denote $y = (y_1, y_2), m = \max\{m_1, m_2\}$. Then we have

$$\begin{aligned} f^{m+n}(y_1) &= f^{(m-m_1)+(m_1+n)}(y_1) = f^{m-m_1}(f^{m_1}(x_1)) = f^m(x_1), \\ f^{m+n}(y_2) &= f^m(x_2), \end{aligned}$$

i.e., $y \in f^{-(m+n)}(f^m(x)) \neq \emptyset$. □

3.5. Corollary. *Let the assumption of* 3.3 *hold. Then there is a connected component $\mathscr{B}$ of (A, f), which is unbounded.*

3.6. Lemma. *Let $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, $(A, f) \in RP(\mathscr{V}_1 \cup \mathscr{V}_2)$. There is a connected component $\mathscr{B}$ of (A, f), which is unbounded.*

Proof. By assumption, $(A, f) \in R(A', f)$, where $(A', f) \in P(\mathscr{V}_1 \cup \mathscr{V}_2)$. Then 3.3 implies that $s_f(x') \neq \infty$ for each $x' \in A'$, thus $s_f(x) \neq \infty$ for each $x \in A$, because $(A, f) \in R(A', f)$. Further, 3.5 yields that there is a connected component $\mathscr{B}'$ of (A', f) such that $\mathscr{B}'$ is unbounded. Consider a retraction endomorphism φ of (A', f) onto (A, f) and let $\mathscr{B}$ be the connected component of (A, f) with $\varphi(\mathscr{B}') \subseteq \mathscr{B}$. Then it is obvious that $\mathscr{B}$ is unbounded. □

3.7. Theorem. *The collection Ant is closed with respect to the operations of* inf *and* sup.

Proof. It suffices to prove that if $\mathscr{V}_1, \mathscr{V}_2 \in Ant$, then $\sup\{\mathscr{V}_1, \mathscr{V}_2\} \in Ant$. Let $\mathscr{V}_1, \mathscr{V}_2 \in \mathscr{V}$. By 1.1, $\sup\{\mathscr{V}_1, \mathscr{V}_2\} = RP(\mathscr{V}_1 \cup \mathscr{V}_2)$. Let $(A, f) \in RP(\mathscr{V}_1 \cup \mathscr{V}_2)$. We have $(A, f) \in R(A', f)$ for some $(A', f) \in P(\mathscr{V}_1 \cup \mathscr{V}_2)$. Then 3.3 implies that $s_f(x') \neq \infty$ for each $x' \in A'$ and since (A, f) is isomorphic to a subalgebra of (A', f), $s_f(x) \neq \infty$ for each $x \in A$ as well. Further, there is a connected component $\mathscr{B}$ of (A, f) which is unbounded in view of 3.6. Hence 3.2 yields that $\sup\{\mathscr{V}_1, \mathscr{V}_2\} \in Ant$. □

In [5], for each ordinal α, a connected monounary algebra $\mathscr{P}_\alpha = (P_\alpha, g)$ was described such that there are distinct elements $p_\alpha, c_\alpha \in P_\alpha$ with the following properties:

(a) $g(p_\alpha) = g(c_\alpha) = c_\alpha$;
(b) if $x \in P_\alpha - \{c_\alpha\}$, then there is $n \in \mathbb{N} \cup \{0\}$ with $g^n(x) = p_\alpha$;
(c) $s_g(p_\alpha) = \alpha$.

4.1. Notation. *For $\alpha \in Ord$ and $n \in \mathbb{N}$ we denote $n_\alpha = (n, \alpha)$. Let $\alpha \in Ord$. Put $\mathbb{N}_\alpha = \{n_\alpha \colon n \in \mathbb{N}\}$. Further, we denote by $\mathscr{A}_\alpha = (A_\alpha, f)$ the monounary algebra such that*

$$\begin{aligned} A_\alpha &= \mathbb{N}_\alpha \cup P_\alpha, \\ f(n_\alpha) &= (n+1)_\alpha \text{ for each } n \in \mathbb{N}, \\ f(x) &= \begin{cases} g(x) & \text{if } x \in P_\alpha - \{c_\alpha\}, \\ 1_\alpha & \text{if } x = c_\alpha. \end{cases} \end{aligned}$$

4.2. Lemma. *If α is an infinite ordinal, then $\mathscr{A}_\alpha$ is unbounded.*

Proof. The assertion follows from the definition of $\mathscr{A}_\alpha$ and from the definition of the degree. □

4.3. Lemma. *If α is an infinite ordinal, then $V(\mathscr{A}_\alpha) \in Ant$.*

Proof. Let α be an infinite ordinal. We will apply 3.2. Suppose that $(A, f) \in V(\mathscr{A}_\alpha)$. By 1.1, $(A, f) \in RP(\mathscr{A}_\alpha)$, i.e., there are a nonempty set I and a monounary algebra (D, f) such that

$$(A, f) \in R(D, f), \tag{1}$$

$$(D, f) = \mathscr{A}_\alpha^{\operatorname{card} I}. \tag{2}$$

Let $x \in A$. By (1), (A, f) is isomorphic to a subalgebra of (D, f), thus the property (B) of $s_f(x)$ implies that there is $y \in D$ with

$$s_f(x) \leqslant s_f(y). \tag{3}$$

Let $y(i)$ be the i–th projection of y into A_α. We have

$$s_f(y) \leqslant s_f(y(i)). \tag{4}$$

Further, 4.1 implies $s_f(y(i)) \neq \infty$, thus (3) and (4) yield

$$s_f(x) \neq \infty \text{ for each } x \in A. \tag{5}$$

Define an element $d \in D$ by putting $d(i) = 1_\alpha$ for each $i \in I$. The algebra $\mathscr{A}_\alpha$ is unbounded, thus for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with

$$f^{-(m+n)}(f^m(1_\alpha)) \neq \emptyset. \tag{6}$$

Since card $f^m(1_\alpha) = 1$ for each $m \in \mathbb{N}$, the relation (6) implies

$$f^{-n}(1_\alpha) \neq \emptyset, \tag{6'}$$

i.e.,

$$f^{-n}(d(i)) \neq \emptyset. \tag{6''}$$

Thus we obtain

$$f^{-n}(d) \neq \emptyset \text{ for each } n \in \mathbb{N}. \tag{7}$$

Then obviously $f^{-(n+1)}(f(d)) \neq \emptyset$ for each $n \in \mathbb{N}$, i.e.,

$$\begin{gathered} \text{there is } d \in D \text{ such that if } n \in \mathbb{N}, \text{ then there is} \\ m(=1) \text{ with } f^{-(n+m)}(f^m(d)) \neq \emptyset. \end{gathered} \tag{8}$$

According to (1), (8) holds in (A, f) as well, thus in view of 3.2, (5) and (8) yield that $\mathscr{V}$ is antiatomic. □

4.4. Lemma. *Let α be an ordinal, $x \in A_\alpha$. Then there is $n \in \mathbb{N} \cup \{0\}$ such that $s_f(x) \leqslant \alpha + n$.*

P r o o f. We have $x \in \mathbb{N}_\alpha \cup P_\alpha$. If $x \in P_\alpha - \{c_\alpha\}$, then the definition of $s_f(x)$ implies $s_f(x) = s_g(x) \leqslant s_g(p_\alpha) = \alpha$. If $x = c_\alpha$, then $s_f(x) = s_f(c_\alpha) = s_f(p_\alpha) + 1 = \alpha + 1$. If $x = k_\alpha, k \in \mathbb{N}$, then $s_f(x) = \alpha + k + 1$. □

4.5. Lemma. *Let α be an ordinal. If $(C, f) \in V(\mathscr{A}_\alpha), x \in C$, then there is $n \in \mathbb{N} \cup \{0\}$ such that $s_f(x) \leqslant \alpha + n$.*

P r o o f. Since $V(\mathscr{A}_\alpha) = RP(\mathscr{A}_\alpha)$, it suffices to show that if $x \in A_\alpha^{\operatorname{card} I}$, $I \neq \emptyset$, then $s_f(x) \leqslant \alpha + n$ for some $n \in \mathbb{N} \cup \{0\}$. Let $x \in A_\alpha^{\operatorname{card} I}$, $I \neq \emptyset$. Then

$$s_f(x) \leqslant s_f(x(i)) \text{ for each } i \in I$$

and 4.4 yields that there is $n \in \mathbb{N} \cup \{0\}$ with $s_f(x) \leqslant \alpha + n$. □

4.6. Lemma. *There is a collection $\mathscr{O}$ of ordinals such that*

(i) *$\mathscr{O}$ is a proper class,*
(ii) *if $\alpha \in \mathscr{O}$, then α is infinite,*
(iii) *if $\alpha, \beta \in \mathscr{O}, \alpha \neq \beta, k \in \mathbb{N}$, then $\alpha \neq \beta + k$.*

P r o o f. The assertion is obvious. $\square$

In what follows let $\mathscr{O}$ be as in 4.6.

4.7. Lemma. *Let $\alpha, \beta \in \mathscr{O}$. If $\alpha < \beta$, then $\mathscr{A}_\beta \notin V(\mathscr{A}_\alpha)$.*

P r o o f. If $\mathscr{A}_\beta \in V(\mathscr{A}_\alpha)$, then according to 4.5 the relation $p_\beta \in \mathscr{A}_\beta$ implies

(1) $$s_f(p_\beta) \leqslant \alpha + n \text{ for some } n \in \mathbb{N} \cup \{0\}.$$

Further, by the definition of $\mathscr{A}_\beta$, we have

(2) $$s_f(p_\beta) = \beta.$$

Then(1), (2) and the assumption yield

(3) $$\alpha < \beta \leqslant \alpha + n.$$

Thus $\beta = \alpha + k$ for some $k \in \mathbb{N}$, which contradicts 4.6. $\square$

4.8. Lemma. *Let $\alpha, \beta \in \mathscr{O}$. If $\alpha > \beta$, then $\mathscr{A}_\beta \notin V(\mathscr{A}_\alpha)$.*

P r o o f. Suppose that $\alpha > \beta$ and $\mathscr{A}_\beta \in V(\mathscr{A}_\alpha)$. Then $\mathscr{A}_\beta \in RP(\mathscr{A}_\alpha)$ and there are $I \neq \emptyset$ and $\mathscr{A}'_\beta = (A'_\beta, f) \cong (A_\beta, f)$ such that $\mathscr{A}'_\beta$ is a retract of $\mathscr{A}_\alpha^{\operatorname{card} I}$. Denote by φ a retraction endomorphism of $\mathscr{A}_\alpha^{\operatorname{card} I}$ onto $\mathscr{A}'_\beta$. Further let $p \in A_\alpha^{\operatorname{card} I}$, where $p(i) = p_\alpha$ for each $i \in I$. Then

(1) $$s_f(p) = \alpha.$$

Since φ is an endomorphism, (1) implies

(2) $$s_f(\varphi(p)) \geqslant s_f(p) = \alpha.$$

Further, $\mathscr{A}'_\beta \cong \mathscr{A}_\beta$, thus 4.4 yields

(3) $$\text{if } x \in \mathscr{A}'_\beta, \text{ then } s_f(x) \leqslant \beta + n \text{ for some } n \in \mathbb{N} \cup \{0\}.$$

By (2) and (3) we obtain

$$\alpha \leqslant s_f(\varphi(p)) \leqslant \beta + n \text{ for some } n \in \mathbb{N} \cup \{0\}. \tag{4}$$

According to (4) the assumption $\alpha > \beta$ implies

$$\beta < \alpha \leqslant \beta + n \text{ for some } n \in \mathbb{N} \cup \{0\},$$

which contradicts the relation $\{\alpha, \beta\} \subseteq \mathscr{O}$. □

4.9. Corollary. *If $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$, then $\mathscr{A}_\beta \notin V(\mathscr{A}_\alpha)$.*

4.10. Corollary. *If $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$, then $V(\mathscr{A}_\beta) \nsubseteq V(\mathscr{A}_\alpha)$.*

4.11. Theorem. *There is a proper class $\mathscr{O}$ of ordinals such that for each $\alpha \in \mathscr{O}$ there exists $\mathscr{W}_\alpha \in Ant$ with the property that if $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$, then $\mathscr{W}_\alpha \nsubseteq \mathscr{W}_\beta$.*

P r o o f. Let $\mathscr{O}$ be as in 4.6. Consider all $\mathscr{A}_\alpha$, $\alpha \in \mathscr{O}$ and put, for $\alpha \in \mathscr{O}$,

$$\mathscr{W}_\alpha = V(\mathscr{A}_\alpha).$$

According to 4.3 and 4.6, if $\alpha \in \mathscr{O}$, then $\mathscr{W}_\alpha \in Ant$. Let $\alpha, \beta \in \mathscr{O}$, $\alpha \neq \beta$. By 4.10 we have $\mathscr{W}_\alpha \nsubseteq \mathscr{W}_\beta$. □

5. Large chain of antiatomic retract varieties

In this section we will apply the algebras $\mathscr{A}_\alpha$ for $\alpha \in \mathscr{O}$ defined in the previous section.

5.1. Notation. *If $\alpha \in \mathscr{O}$, then we denote*

$$\mathscr{V}_\alpha = V(\{\mathscr{A}_\beta\colon\ \beta \leqslant \alpha\}).$$

5.2. Lemma. *If $\alpha, \beta \in \mathscr{O}, \beta < \alpha$, then $\mathscr{V}_\beta \subseteq \mathscr{V}_\alpha$.*

P r o o f. The assertion follows from 5.1. □

5.3. Lemma. *If $\alpha \in \mathscr{O}$, then $\mathscr{V}_\alpha \in Ant$.*

Proof. We shall prove the assertion by means of 3.2. Let $\alpha \in \mathscr{O}, (A, f) \in \mathscr{V}_\alpha$. By 1.1,

$$(A, f) \in RP(\{\mathscr{A}_\beta\colon \beta \leqslant \alpha\}),$$

i.e., there are a nonempty set I and monounary algebras (D, f) and (B_i, f) for each $i \in I$ such that

$$(A, f) \in R(D, f), \tag{1}$$

$$(D, f) = \prod_{i \in I} (B_i, f), \tag{2}$$

$$\{(B_i, f)\colon i \in I\} \subseteq \{\mathscr{A}_\beta\colon \beta \leqslant \alpha\}. \tag{3}$$

Let $x \in A$. Since (A, f) is isomorphic to a subalgebra of (D, f), by (B) there is $y \in D$ with $s_f(x) \leqslant s_f(y)$. Further, (A) yields that $s_f(y) \leqslant s_f(y(i))$ for each $i \in I$. According to 3.2, (B_i, f) is unbounded for $i \in I$, hence $s_f(y(i)) \neq \infty$, thus $s_f(x) \neq \infty$, too.

The relation (3) implies that if $i \in I$, then $(B_i, f) = \mathscr{A}_{\beta_i}$ for some $\beta_i \in \mathscr{O}$. Consider the element $d \in D$ such that $d(i) = 1_{\beta_i}$ for each $i \in I$. The remaining part of the proof is analogous as in 4.3. $\square$

5.4. Lemma. *Let $\alpha \in \mathscr{O}$, $x \in D, (D, f) \in \mathscr{V}_\alpha$. Then there is $n \in \mathbb{N} \cup \{0\}$ such that $s_f(x) \leqslant \alpha + n$.*

Proof. It suffices to show that if $I \neq \emptyset$ and (D, f) satisfies (2) and (3) of 5.3, then the assertion is valid. In the case under consideration we have

$$s_f(x) \leqslant s_f(x(i)) \text{ for each } i \in I.$$

By 4.4, $s_f(x(i)) \leqslant \beta_i + n$ for some $n \in \mathbb{N} \cup \{0\}$, where $\beta_i \leqslant \alpha$, hence we obtain $s_f(x) \leqslant \alpha + n$ for some $n \in \mathbb{N} \cup \{0\}$. $\square$

5.5. Lemma. *Let $\alpha, \beta \in \mathscr{O}$. If $\beta < \alpha$, then $\mathscr{A}_\alpha \notin \mathscr{V}_\beta$.*

Proof. Suppose that $\beta < \alpha$ and $\mathscr{A}_\alpha \in \mathscr{V}_\beta$. By the definition of $\mathscr{A}_\alpha$ we have

$$s_f(p_\alpha) = \alpha. \tag{1}$$

Further, 5.4 implies

$$s_f(p_\alpha) \leqslant \beta + n \text{ for some } n \in \mathbb{N} \cup \{0\}. \tag{2}$$

Thus $\beta < \alpha \leqslant \beta + n$, $\alpha = \beta + k$ for some $k \in \mathbb{N}$, which contradicts 4.6. $\square$

5.6. Corollary. *If $\alpha, \beta \in \mathscr{O}, \beta < \alpha$, then $\mathscr{V}_\beta \subsetneqq \mathscr{V}_\alpha$.*

P r o o f. It is a consequence of 5.2 and 5.5. □

5.7. Theorem. *There is a proper class $\mathscr{O}$ of ordinals such that for each $\alpha \in \mathscr{O}$ there exists $\mathscr{V}_\alpha \in Ant$ with the property that if $\alpha, \beta \in \mathscr{O}$, $\beta < \alpha$, then $\mathscr{V}_\beta \subsetneqq \mathscr{V}_\alpha$.*

P r o o f. The assertion follows from 5.3 and 5.6. □

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