SOLUTION OF THE NEUMANN PROBLEM FOR THE LAPLACE EQUATION

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Abstract. For fairly general open sets it is shown that we can express a solution of the Neumann problem for the Laplace equation in the form of a single layer potential of a signed measure which is given by a concrete series. If the open set is simply connected and bounded then the solution of the Dirichlet problem is the double layer potential with a density given by a similar series.

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Suppose that $G \subset \mathbb{R}^m$ $(m \geqslant 2)$ is an open set with a compact boundary ∂G. If h is a harmonic function on G such that

$$\int_H |\operatorname{grad} h| \,\mathrm{d}\mathscr{H}_m < \infty$$

for all bounded open subsets H of G we define the weak normal derivative $N^G h$ of h as a distribution

$$\langle \varphi, N^G h\rangle = \int_G \operatorname{grad} \varphi \cdot \operatorname{grad} h \,\mathrm{d}\mathscr{H}_m$$

for $\varphi \in \mathscr{D}$ (= the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$). Here $\mathscr{H}_k$ is the k-dimensional Hausdorff measure normalized so that $\mathscr{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$. We formulate the Neumann problem for the Laplace equation with a boundary condition $\mu \in \mathscr{C}'$ (= the Banach space of all finite signed Borel measures with support in ∂G with the total variation as a norm) as follows:

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determine a harmonic function h on G for which $N^G h = \mu$. We wish to find the function h in the form of the single layer potential

$$\mathscr{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y)\,\mathrm{d}\nu(y),$$

where $\nu \in \mathscr{C}'$,

$$\begin{aligned} h_x(y) = (m-2)^{-1}A^{-1}|x-y|^{2-m} &\quad \text{for } m > 2,\\ A^{-1}\log|x-y|^{-1} &\quad \text{for } m = 2, \end{aligned}$$

A is the area of the unit sphere in $\mathbb{R}^m$. The single layer potential $\mathscr{U}\nu$ is a harmonic function in G for which the weak normal derivative $N^G\mathscr{U}\nu$ has sense. The operator $N^G\mathscr{U} : \nu \mapsto N^G\mathscr{U}\nu$ is a bounded linear operator on $\mathscr{C}'$ if and only if $V^G < \infty$, where

$$V^G = \sup_{x\in\partial G} v^G(x),$$
$$v^G(x) = \sup\left\{ \int_G \operatorname{grad}\varphi\cdot\operatorname{grad}h_x\,\mathrm{d}\mathscr{H}_m\,;\ \varphi\in\mathscr{D}, |\varphi|\leqslant 1, \operatorname{spt}\varphi\subset\mathbb{R}^m-\{x\}\right\}$$

(see [9]). There are more geometrical characterizations of $v^G(x)$ in [9] which ensure $V^G < \infty$ for G convex or for G with $\partial G \subset \bigcup_{i=1}^{k} L_i$, where L_i are $(m-1)$-dimensional Ljapunov surfaces i.e. of class $C^{1+\alpha}$ (see [16]).

If $z \in \mathbb{R}^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x\in\mathbb{R}^m\,;\ (x-z)\cdot\theta > 0\}$ has m-dimensional density zero at z then $n^G(z) = \theta$ is termed *the interior normal* of G at z in Federer's sense. If there is no interior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$. The set $\{y\in\mathbb{R}^m\,;\ |n^G(y)| > 0\}$ is called the reduced boundary of G and will be denoted by $\widehat{\partial}G$.

If G has a finite perimeter (which is fulfilled if $V^G < \infty$) then $\mathscr{H}_{m-1}(\widehat{\partial}G) < \infty$ and

$$v^G(x) = \int_{\widehat{\partial}G} |n^G(y)\cdot\operatorname{grad}h_x(y)|\,\mathrm{d}\mathscr{H}_{m-1}(y)$$

for each $x\in\mathbb{R}^m$. Throughout the paper we shall assume that $V^G < \infty$.

Denote $C = \mathbb{R}^m - \operatorname{cl}G$ and suppose for a while that $\partial C = \partial G$. For $x\in\mathbb{R}^m$, $f\in\mathscr{C}$, where $\mathscr{C}$ is the space of all bounded continuous functions on ∂G equipped with the maximum norm, we may define

$$W^G f(x) = \mathrm{d}_G(x)f(x) - \int_{\partial G} f(y)n^G(y)\cdot\operatorname{grad}h_x(y)\,\mathrm{d}\mathscr{H}_{m-1}(y),$$

where

$$\mathrm{d}_G(x) = \lim_{r\to 0_+} \frac{\mathscr{H}_m\big(\mathscr{U}(x;r)\cap G\big)}{\mathscr{H}_m\big(\mathscr{U}(x;r)\big)}$$

is the m-dimensional density of G at the point x and $\mathscr{U}(x;r) = \{y \in \mathbb{R}^m\,;\ |x-y| < r\}$. (If $V^G < \infty$ then there is $d_G(x)$ for all $x \in \mathbb{R}^m$ (see [9], Lemma 2.9).) The double layer potential $W^G f$ is a function harmonic on $\mathbb{R}^m - \mathrm{cl}\, G$ and continuous on ∂G. Besides that $W^G : f \mapsto W^G f$ is a bounded operator on $\mathscr{C}$ and $N^G\mathscr{U}$ is the dual operator of W^G. If $W^G f = g$ on ∂G then $W^G f$ is a solution of the Dirichlet problem on C with the boundary condition g (see [9], Theorem 2.19).

If we denote $T^G = 2W^G - I$, where I is the identity operator, then the Dirichlet problem for C and the Neumann problem for G lead to the dual equations

$$(I + T^G)f = 2g, \tag{1}$$

$$(I + T^G)^*\nu = 2\mu. \tag{2}$$

Here L^* denotes the dual operator to the operator L.

If L is a bounded linear operator on the Banach space X we denote by $\|L\|_{\mathrm{ess}}$ the essential norm of L, i.e. the distance of L from the space of all compact linear operators on X. If $\|T^G\|_{\mathrm{ess}} < 1$ then G has a finite number of components and the equation $(I + T^G)^*\nu = 2\mu$ has a solution if and only if $\mu(\partial H) = 0$ for each bounded component H of G. The equation $(I+T^G)f = 2g$ has a solution for each $g \in \mathscr{C}$ if and only if G is unbounded and connected. (See [9].) It is well-known that this condition is fulfilled for sets with a smooth boundary (of class $C^{1+\alpha}$) and for convex sets (see [9], [12]). J. Radon proved this condition for a set with bounded rotation in the plane (particularly for a set with a piecewise smooth boundary without cusps) (see [21], [22]). But this condition does not hold even for rectangular domains (i.e. formed by rectangular parallelepipeds) in $\mathbb{R}^3$ (see [10]). If $G \subset \mathbb{R}^3$ is a rectangular domain then there is a norm $\|\!|\ \|\!|$ on $\mathscr{C}$ equivalent to the maximum norm such that $\big|\|\!|T^G\|\!|\big|_{\mathrm{ess}} < 1$ (see [10], [1]). This condition is equivalent to

$$r_{\mathrm{ess}}(T^G) < 1, \tag{3}$$

where the essential radius of the bounded linear operator L on the Banach space X is defined by

$$r_{\mathrm{ess}}(L) = \liminf_{n\to\infty} \big(\|L^n\|_{\mathrm{ess}}\big)^{\frac{1}{n}}$$

(see [4]).

If X is a real Banach space we denote by $^\wedge X$ the complexification of X. If L is a linear operator on X we extend L to $^\wedge X$ by $L(x + \mathrm{i}y) = Lx + \mathrm{i}Ly$. According to

[26], Chapter IX, Theorem 2.1 and Theorem 1.3 the operator $\lambda I - T^G$ is a Fredholm operator on ${}^{\wedge}\mathscr{C}$ for all complex λ with $|\lambda| \geqslant 1$ if and only if (3) holds.

A. Rathsfeld showed in [23], [24] that (3) holds for a polyhedral cone in $\mathbb{R}^3$. (Compare with the analogical result in [7].)The condition (3) holds even for $G \subset \mathbb{R}^3$ with a piecewise smooth boundary (see [14]).

It is shown in this article that if T^G is quasicompact (i.e. $r_{\mathrm{ess}}(T^G) < 1$) then $\mathrm{cl}\, G$ has a finite number of components. The Neumann problem for G with the boundary condition $\mu \in \mathscr{C}'$ has a solution if and only if $\mu(\partial H) = 0$ for each bounded component H of $\mathrm{cl}\, G$. We can take this solution in the form of the single layer potential $\mathscr{U}\nu$ where $\nu \in \mathscr{C}'$ is a solution of the equation $(I + T^G)^*\nu = 2\mu$. The equation $(I + T^G)f = 2g$ has a solution for each $g \in \mathscr{C}$ if and only if $\mathrm{cl}\, G$ is unbounded and connected.

But how to calculate a solution of the equation (1) or (2)? If G is convex then the series

$$\sum_{n=0}^{\infty} \big[(-T^G)^*\big]^n (2\mu) \tag{4}$$

represents a solution of (2) for each $\mu \in \mathscr{C}'$ such that $\mu(\partial G) = 0$.

The attempt to justify the convergence of the series obtained from the equation (1) led C. Neumann to his investigation [17]–[19] of contractivity (for convex domains) of the operator T^G called by him the operator of the arithmetical mean. Neumann's method led to further investigation of domains with a smooth boundary by J. Plemelj (cf. [20]). His approach forms the basis of this paper.

The aim of this article is to prove that if G satisfies (3) then a solution of the Neumann problem for G with the boundary condition $\mu \in \mathscr{C}'$ can be taken in the form of the single layer potential $\mathscr{U}\nu$ where ν is given by the series

$$\mu + \sum_{n=0}^{\infty} \big[(-T^G)^*\big]^n \big[I - (T^G)^*\big]\mu.$$

If $\mathbb{R}^m - G$ is unbounded and connected then we can take ν even in the form of the series (4). This condition is necessary for the convergence of the series (4) for each $\mu \in \mathscr{C}'$ for which there is a solution of the Neumann problem with the boundary condition μ. If $\partial C = \partial G$ and $\mathrm{cl}\, G$ is unbounded and connected then a solution of the Dirichlet problem for C with the boundary condition $g \in \mathscr{C}$ can be taken in the form of the double layer potential $W^G f$ where

$$f = g + \sum_{n=0}^{\infty} \big(-T^G\big)^n (I - T^G) g.$$

Lemma 1. *If* (3) *holds then the set* $\mathscr{I}$ *of all isolated points of* ∂G *is finite and*

$$0 < \inf_{x \in \partial G - \mathscr{I}} \mathrm{d}_G(X) \leqslant \sup_{x \in \partial G - \mathscr{I}} \mathrm{d}_G(X) < 1.$$

Proof. (See proof of Theorem 4.1 in [9].) Since T^G is quasicompact there are a natural number n and a compact linear operator K on $\mathscr{C}$ such that

$$\|\big(T^G\big)^n + K\| < 1. \tag{5}$$

By the Radon theorem K can be arbitrarily closely approximated by finite dimensional operators of the form

$$\widetilde{K} f = \sum_{k=1}^{q} \langle f, \nu_k \rangle \varphi_k$$

with $\varphi_k \in \mathscr{C}$ and $\nu_k \in \mathscr{C}'$ (see [9], pp. 102–103; compare Chapter V in [25]). Clearly, there is K of the form

$$K f = \sum_{k=1}^{q_1} \langle f, \nu_k \rangle \varphi_k + \sum_{k=1}^{q_2} \psi_k f(y_k)$$

where $M = \{y_1, \dots, y_{q_2}\} \subset \partial G$, $\varphi_k \in \mathscr{C}$, $\psi_k \in \mathscr{C}$, $\nu_k \in \mathscr{C}'$, ν_k does not charge single point sets and (5) is true.

Denote

$$k_1(x, y) = -2 n^G(y) \cdot \operatorname{grad} h_x(y)$$

for x, $y \in \partial G$. For fixed $x \in \partial G$ and a natural number p we define $k_p(x, y)$ by the recurrent formula

$$k_{p+1}(x, y) = \int_{\partial G} k_1(x, z) k_p(z, y) \, \mathrm{d}\mathscr{H}_{m-1}(z).$$

By the inductive method we prove that for a fixed x the function $k_p(x, y)$ is defined for $\mathscr{H}_{m-1}$-a.a. $y \in \partial G$, vanishes outside $\widehat{\partial} G$ and

$$\int_{\partial G} |k_p(x, y)| \, \mathrm{d}\mathscr{H}_{m-1}(y) \leqslant 2^p (V^G)^p.$$

Since $\big(2\, \mathrm{d}_G(x) - 1\big) = 0$ on $\widehat{\partial} G$ we obtain by the inductive method

$$\begin{aligned}
(T^G)^p f(x) = {} & \big(2\, \mathrm{d}_G(x) - 1\big)^p f(x) + \big(2\, \mathrm{d}_G(x) - 1\big)^{p-1} \int_{\partial G} k_1(x, y) f(y) \, \mathrm{d}\mathscr{H}_{m-1}(y) \\
& + \big(2\, \mathrm{d}_G(x) - 1\big)^{p-2} \int_{\partial G} k_2(x, y) f(y) \, \mathrm{d}\mathscr{H}_{m-1}(y) + \dots \\
& + \big(2\, \mathrm{d}_G(x) - 1\big) \int_{\partial G} k_{p-1}(x, y) f(y) \, \mathrm{d}\mathscr{H}_{m-1}(y) \\
& + \int_{\partial G} k_p(x, y) f(y) \, \mathrm{d}\mathscr{H}_{m-1}(y).
\end{aligned}$$

Put

$$k(x,y)=\sum_{j=1}^{n}\bigl(2\,\mathrm{d}_G(x)-1\bigr)^{n-j}k_j(x,y).$$

Then

$$\bigl(T^G\bigr)^n f(x)=\bigl(2\,\mathrm{d}_G(x)-1\bigr)^n f(x)+\int_{\partial G}k(x,y)f(y)\,\mathrm{d}\mathscr{H}_{m-1}(y).$$

Denote by λ_x the measure

$$\int f\,\mathrm{d}\lambda_x=\bigl(T^G\bigr)^n f(x).$$

Then for $x\in\partial G-M$

$$\begin{aligned}\Bigl\|\lambda_x+\sum_{k=1}^{q_1}\varphi_k(x)\nu_k\Bigr\| &\leqslant \|\lambda_x+\sum_{k=1}^{q_1}\varphi_k(x)\nu_k\|+\sum_{k=1}^{q_2}|\psi_k(x)|\\ &=\Bigl\|\lambda_x+\sum_{k=1}^{q_1}\varphi_k(x)\nu_k+\sum_{k=1}^{q_2}\psi_k(x)\delta_{y^k}\Bigr\|\\ &=\sup\bigl\{|(T^G)^n f(x)+Kf(x)|;\ f\in\mathscr{C},|f|\leqslant 1\bigr\}\leqslant\|\bigl(T^G\bigr)^n+K\|.\end{aligned}$$

Put

$$\widetilde{K}f(y)=\sum_{k=1}^{q_1}\varphi_k(y)\langle f,\nu_k\rangle.$$

Then $\bigl(T^G\bigr)^n+\widetilde{K}$ is a bounded operator on $\mathscr{C}$. Let now $\varphi\in\mathscr{C}$, $|\varphi|\leqslant 1$. Since for $x\in\partial G-M$

$$|(T^G)^n\varphi(x)+\widetilde{K}\varphi(x)|\leqslant\Bigl\|\lambda_x+\sum_{k=1}^{q_1}\varphi_k(x)\nu_k\Bigr\|\leqslant\|(T^G)^n+K\|$$

the continuity of the function $(T^G)^n\varphi+\widetilde{K}\varphi$ yields $\|(T^G)^n\varphi(x)+\widetilde{K}\varphi(x)\|\leqslant\|(T^G)^n+K\|$ for $x\in\mathrm{cl}(\partial G-M)$. For fixed $x\in\mathrm{cl}(\partial G-M)$ and a natural number k put $\varphi_k(y)=\max(0,1-k|y-x|)$. Then we obtain from (5) that $|2\,\mathrm{d}_G(x)-1|^n=\lim\limits_{k\to\infty}|(T^G)^n\varphi_k(x)+\widetilde{K}\varphi_k(x)|\leqslant\|(T^G)^n+K\|<1$. Since $\partial G-\mathscr{I}\subset\mathrm{cl}(\partial G-M)$ we have $\mathscr{I}\subset M$, $\mathscr{I}$ is finite and the inequality in the lemma holds. □

Lemma 2. *If* $r_{\mathrm{ess}}(T^G)<1$ *then* $\mathscr{H}_{m-1}(\partial G)<\infty$, $\mathscr{H}_{m-1}(\partial G-\widehat{\partial}G)=0$.

P r o o f. Since G has a finite perimeter and $0<\mathrm{d}_G(x)<1$ for $\mathscr{H}_m$-a.a. $x\in\partial G$ by Lemma 1, we obtain $\mathscr{H}_{m-1}(\widehat{\partial}G)<\infty$ and $\mathscr{H}_{m-1}(\partial G-\widehat{\partial}G)=0$ by the Gauss-Green theorem (see [3], Theorem 4.5.6). □

Note 1. Denote $\widetilde{G} = \operatorname{int}\operatorname{cl} \mathrm{G}$. Then $\mathscr{H}_m(\widetilde{G} - G) = 0$, $\partial\widetilde{G} = \partial C$, $V^{\widetilde{G}} < \infty$, $N^{\widetilde{G}} = N^G$. If $\nu \in \mathscr{C}'$, $\nu(M) = 0$ for $M \subset \partial G - \partial\widetilde{G}$ then $N^G\mathscr{U}\nu(M) = \nu(M)$ for $M \subset \partial G - \partial\widetilde{G}$. If $r_{\mathrm{ess}}(T^G) < 1$ then we obtain $r_{\mathrm{ess}}(T^{\widetilde{G}}) < 1$ because ∂G and $\partial\widetilde{G}$ differ only at finitely many isolated points of ∂G by Lemma 1. So, throughout the rest of the paper we will assume that $\partial G = \partial C$.

Lemma 3. *If W^G is Fredholm then $\operatorname{cl} G$ has a finite number of components.*

Proof. Suppose the opposite. Then we are going to construct such a sequence $\{A_j\}$ of nonempty closed subsets of $\operatorname{cl} G$ that $\operatorname{cl} G - A_j$ is closed, $A_{j+1} \subsetneqq A_j$ and A_j has infinitely many components. Put $A_1 = \operatorname{cl} G$. For a given A_j we construct A_{j+1} in the following way. Since A_j is not connected there are nonempty closed disjoint sets C, D such that $C \cup D = A_j$. If H is a component of A_j then $C \cap H$, $H \cap D$ are closed sets. Since H is connected, necessarily $C \cap H = \emptyset$ or $H \cap D = \emptyset$ and thus either $H \subset C$ or $H \subset D$. Now we denote by A_{j+1} one of the sets C, D which has infinitely many components.

If there is a natural number i such that A_i is bounded we put $B_j = A_j$ for $j \geqslant i$. If A_j is unbounded for each j we put $i = 1$, $B_j = \operatorname{cl} G - A_j$. Now we choose for every $j \geqslant i$ a function $\varphi_j \in \mathscr{D}$ such that $\varphi_j = 1$ on a neighbourhood of B_j and $\varphi_j = 0$ on a neighbourhood of $\operatorname{cl} G - B_j$. If $\nu \in \mathscr{C}'$ then

$$(N^G\mathscr{U}\nu)(\partial B_j) = \langle \varphi_j, N^G\mathscr{U}\nu\rangle = \int_G \operatorname{grad}\varphi_j \cdot \operatorname{grad}\mathscr{U}\nu = 0.$$

So $N^G\mathscr{U}(\mathscr{C}')$ has an infinite codimension in $\mathscr{C}'$. Since $N^G\mathscr{U}$ is the dual operator of W^G the operator $N^G\mathscr{U}$ is Fredholm, too, by [26], Chapter VII, Theorem 3.5. This is a contradiction. □

Note 2. If $r_{\mathrm{ess}}(T^G) < 1$ then $r_{\mathrm{ess}}(T^C) < 1$ because $T^C = -T^G$. So, if $r_{\mathrm{ess}}(T^G) < 1$ then $\operatorname{cl} G$ and $\mathbb{R}^m - G$ have a finite number of components by Lemma 3 and [26], Chapter IX, Theorem 2.1 and Theorem 1.3.

Definition. We shall denote by $\mathscr{C}'_c$ the subspace of those $\mu \in \mathscr{C}'$ for which there exists a (finite) continuous function $\mathscr{U}_c\mu$ on $\mathbb{R}^m$ such that $\mathscr{U}_c\mu = \mathscr{U}\mu$ on $\mathbb{R}^m - \partial G$.

Lemma 4. *Let p be a positive integer and λ a complex number with $|\lambda| > r_{\mathrm{ess}}(T^G)$. Then any $\mu \in {}^{\wedge}\mathscr{C}'$ satisfying the homogeneous equation*

$$\big[(T^G)^* + \lambda I\big]^p \mu = 0$$

necessarily belongs to ${}^{\wedge}\mathscr{C}'_c$.

Proof. The lemma is an easy generalization of [9], Theorem 4.10 and we can obtain it by repeating all reasonings in [9], §4. □

Notation. Let us define a function θ on $\mathbb{R}^m$ as follows:

$$\theta(x) = \exp\left(|x|^2 - 1\right)^{-1} \quad \text{for } |x| < 1,$$
$$\theta(x) = 0 \quad \text{for } |x| \geqslant 1.$$

For $\delta > 0$ put

$$\theta_\delta(x) = h_\delta \theta(x/\delta)$$

with $h_\delta \in \mathbb{R}$ chosen so that

$$\int_{\mathbb{R}^m} \theta_\delta(x) \, \mathrm{d}\mathscr{H}_m(x) = 1.$$

Clearly, $\theta_\delta \in \mathscr{D}$ for each δ.

If f is locally integrable over $\mathbb{R}^m$ we denote

$$R_\delta f(x) = \int_{\mathbb{R}^m} f(y)\theta_\delta(x-y) \, \mathrm{d}\mathscr{H}_m(y), \quad x \in \mathbb{R}^m.$$

Then $R_\delta f \in \mathscr{D}$. If $|f(y)| \leqslant \beta$ holds for $\mathscr{H}_m$-almost all $y \in \mathbb{R}^m$ then the inequality

$$|R_\delta f(x)| \leqslant \beta$$

is true for any $x \in \mathbb{R}^m$. If f is continuous then $R_\delta f$ converges locally uniformly to f for $\delta \to 0_+$.

Finally, for each $\varepsilon > 0$ let

$$B^\varepsilon = \left\{x \in \mathbb{R}^m ;\ \operatorname{dist}(x, \partial G) > \varepsilon\right\}.$$

Lemma 5. *Suppose that $\mu \in \mathscr{C}'$ and $\varepsilon > 0$. Then*

$$\lim_{\delta \to 0_+} R_\delta \mathscr{U}\mu = \mathscr{U}\mu$$

holds quasi - everywhere in $\mathbb{R}^m$ and for each $\delta \in (0, \varepsilon)$ we have $R_\delta \mathscr{U}\mu = \mathscr{U}\mu$ on B^ε.

Proof. See [15], proof of Lemma 22. □

Lemma 6. *Suppose $\mathscr{H}_m(\partial G) = 0$. Let $\mu \in \mathscr{C}'_c$. In the case $m = 2$ suppose moreover that $\mu(\mathbb{R}^m) = 0$. Then*

$$\sup_{\delta \in (0,1)} \int_{\mathbb{R}^m} |\operatorname{grad} R_\delta \mathscr{U}\mu|^2 \, \mathrm{d}\mathscr{H}_m < \infty,$$
$$\int_{\mathbb{R}^m} |\operatorname{grad} \mathscr{U}\mu|^2 \, \mathrm{d}\mathscr{H}_m < \infty.$$

P r o o f. Since

$$\lim_{|x|\to\infty} |\mathscr{U}\mu(x)| = 0$$

there is $\beta \in \mathbb{R}^1$ such that $|\mathscr{U}_c\mu| \leqslant \beta$. Fix $R > 1$ such that $\partial G \subset \mathscr{U}(0;R)$. Suppose $r > 2R$, $\delta \in (0,1)$. By the Gauss-Green theorem we get

$$\begin{aligned} (6)\qquad & \int_{\partial\mathscr{U}(0;r)} R_\delta\mathscr{U}\mu(z)\big(-n^{\mathscr{U}(0;r)}(z)\big)\cdot \operatorname{grad}\big(R_\delta\mathscr{U}\mu(z)\big)\,\mathrm{d}\mathscr{H}_{m-1}(z) \\ &= \int_{\mathscr{U}(0;r)} |\operatorname{grad}\big(R_\delta\mathscr{U}\mu(x)\big)|^2\,\mathrm{d}\mathscr{H}_m(x) \\ &\quad + \int_{\mathscr{U}(0;r)} \big(R_\delta\mathscr{U}\mu(x)\big)\Delta\big(R_\delta\mathscr{U}\mu(x)\big)\,\mathrm{d}\mathscr{H}_m(x). \end{aligned}$$

Let $\varphi \in \mathscr{D}$ satisfy $|\varphi| \leqslant 1$ on $\mathbb{R}^m$ and $\varphi = 1$ on $\mathscr{U}(0;2R)$. By Lemma 5 the function $R_\delta\mathscr{U}\mu$ is harmonic on $\mathbb{R}^m - \mathscr{U}(0;2R)$ and we conclude that

$$\begin{aligned} (7)\qquad & \int_{\mathscr{U}(0;r)} \big(R_\delta\mathscr{U}\mu(x)\big)\Delta\big(R_\delta\mathscr{U}\mu(x)\big)\,\mathrm{d}\mathscr{H}_m(x) \\ &= \int_{\mathbb{R}^m} \varphi(x)\big(R_\delta\mathscr{U}\mu(x)\big)\Delta\big(R_\delta\mathscr{U}\mu(x)\big)\,\mathrm{d}\mathscr{H}_m(x). \end{aligned}$$

It is well-known that $\Delta\mathscr{U}\mu = -\mu$ in the sense of distributions. Since $R_\delta\mathscr{U}\mu = \theta_\delta * (\mathscr{U}\mu)$ is the convolution of the functions θ_δ and $\mathscr{U}\mu$ we have $\Delta(R_\delta\mathscr{U}\mu) = \theta_\delta * (\Delta\mathscr{U}\mu) = \theta_\delta * (-\mu)$ in the sense of distributions (compare [27]). Since $\varphi(R_\delta\mathscr{U}\mu) \in \mathscr{D}$ we have

$$\begin{aligned} (8)\qquad & \int_{\mathbb{R}^m} \varphi(x)\big(R_\delta\mathscr{U}\mu(x)\big)\Delta\big(R_\delta\mathscr{U}\mu(x)\big)\,\mathrm{d}\mathscr{H}_m(x) \\ &= -\int_{\mathbb{R}^m} R_\delta(\varphi R_\delta\mathscr{U}\mu)(x)\,\mathrm{d}\mu(x). \end{aligned}$$

Since $|R_\delta\mathscr{U}\mu| \leqslant \beta$, because $|\mathscr{U}\mu| \leqslant \beta$ on $\mathbb{R}^m - \partial G$ and $\mathscr{H}_m(\partial G) = 0$, we get from (6), (7) and (8) the estimate

$$\begin{aligned} \int_{\mathscr{U}(0;r)} |\operatorname{grad} R_\delta\mathscr{U}\mu(x)|^2\,\mathrm{d}\mathscr{H}_m &\leqslant \beta\|\mu\| + \int_{\partial\mathscr{U}(0;r)} |R_\delta\mathscr{U}\mu|\,|\operatorname{grad} R_\delta\mathscr{U}\mu|\,\mathrm{d}\mathscr{H}_{m-1}(z) \\ &= \beta\|\mu\| + \int_{\partial\mathscr{U}(0;r)} |\mathscr{U}\mu|\,|\operatorname{grad}\mathscr{U}\mu|\,\mathrm{d}\mathscr{H}_{m-1} \\ &\leqslant \beta\|\mu\| + \beta\frac{1}{A}\frac{\|\mu\|}{(r-R)^{m-1}}Ar^{m-1} \leqslant 2^m\beta\|\mu\| \end{aligned}$$

by Lemma 5. Hence

$$(9)\qquad \int_{\mathbb{R}^m} |\operatorname{grad} R_\delta\mathscr{U}\mu|^2\,\mathrm{d}\mathscr{H}_m \leqslant 2^m\beta\|\mu\|.$$

Lemma 5 yields

$$\lim_{\delta\to 0_+} \operatorname{grad} R_\delta \mathscr{U}\mu(x) = \operatorname{grad}\mathscr{U}\mu(x)$$

whenever $x \in \mathbb{R}^m - \partial G$. Since $\mathscr{H}_m(\partial G) = 0$, Fatou's lemma may be applied to assert $\int_{\mathbb{R}^m} |\operatorname{grad}\mathscr{U}\mu|^2 \leqslant 2^m\beta\|\mu\|$. □

Lemma 7. *Suppose $\mathscr{H}_m(\partial G) = 0$. Let ν_1, $\nu_2 \in \mathscr{C}_c'$. In the case $m = 2$ suppose moreover that $\nu_i(\mathbb{R}^m) = 0$ for $i = 1, 2$. Then*

$$\int_{\partial G} \mathscr{U}_c\nu_1 \, \mathrm{d}N^G\mathscr{U}\nu_2 = \int_G \operatorname{grad}\mathscr{U}\nu_1 \cdot \operatorname{grad}\mathscr{U}\nu_2 \, \mathrm{d}\mathscr{H}_m.$$

P r o o f. (Compare with [15].) Let ψ be an infinitely differentiable function in $\mathbb{R}^1$, $0 \leqslant \psi \leqslant 1$, $\psi(t) = 1$ for $t \in \langle 0, 1\rangle$ and $\psi(t) = 0$ for $t \in (2, \infty)$. For $\delta > 0$, $x \in \mathbb{R}^m$ put

$$\begin{aligned}\psi_\delta(x) &= \psi(\delta|x|),\\ \varphi_\delta(x) &= \psi_\delta(x)(R_\delta\mathscr{U}_c\nu_1)(x).\end{aligned}$$

Since $\mathscr{U}_c\nu_1$ is continuous, φ_δ converge to $\mathscr{U}_c\nu_1$ uniformly on ∂G for $\delta \to 0_+$. Since $\varphi_\delta \in \mathscr{D}$ we have

(10)
$$\int_{\partial G} \mathscr{U}_c\nu_1 \, \mathrm{d}N^G\mathscr{U}\nu_2 = \lim_{\delta\to 0_+} \int_{\partial G} \varphi_\delta \, \mathrm{d}N^G\mathscr{U}\nu_2 = \lim_{\delta\to 0_+} \int_G \operatorname{grad}\varphi_\delta \cdot \operatorname{grad}\mathscr{U}\nu_2 \, \mathrm{d}\mathscr{H}_m.$$

We are going to prove

$$\int_{\mathbb{R}^m} |\operatorname{grad}\varphi_\delta|^2 \, \mathrm{d}\mathscr{H}_m \leqslant K \quad \text{for } \delta \in (0, \delta_0). \tag{11}$$

Choose $\delta_0 \in (0, 1/2)$ such that $\partial G \subset \mathscr{U}\big(0; 1/(2\delta_0)\big)$. Let $\delta \in (0, \delta_0)$. Denote by χ the characteristic function of the set $\mathscr{U}(0; 2/\delta) - \mathscr{U}(0; 1/\delta)$. Since $R_\delta\mathscr{U}_c\nu_1 = \mathscr{U}\nu_1$ on $\mathbb{R}^m - \mathscr{U}(0; 1/\delta_0)$ by Lemma 5 we have

$$\begin{aligned}
\int_{\mathbb{R}^m} &|\operatorname{grad}\varphi_\delta|^2 \, \mathrm{d}\mathscr{H}_m = \int_{\mathbb{R}^m} |\psi_\delta \operatorname{grad}(R_\delta\mathscr{U}_c\nu_1) + (R_\delta\mathscr{U}_c\nu_1)\operatorname{grad}\psi_\delta|^2 \, \mathrm{d}\mathscr{H}_m\\
&\leqslant \int_{\mathbb{R}^m} \big[|\operatorname{grad} R_\delta\mathscr{U}_c\nu_1| + |\mathscr{U}\nu_1|\chi \sup|\psi'|\delta\big]^2 \, \mathrm{d}\mathscr{H}_m\\
&\leqslant \int_{\mathbb{R}^m} |\operatorname{grad} R_\delta\mathscr{U}\nu_1|^2 \, \mathrm{d}\mathscr{H}_m\\
&\quad + \int_{\mathscr{U}(0;2/\delta)-\mathscr{U}(0;1/\delta)} \big[(\sup|\psi'|)^2\delta^2|\mathscr{U}\nu_1|^2 + 2|\mathscr{U}\nu_1|\delta|\operatorname{grad}\mathscr{U}\nu_1|\sup|\psi'|\big] \, \mathrm{d}\mathscr{H}_m.
\end{aligned}$$

Since there is a positive constant L such that

$$|\mathscr{U}\nu_1(x)| \leqslant \frac{L}{|x|^{m-2}},$$
$$|\operatorname{grad}\mathscr{U}\nu_1(x)| \leqslant \frac{L}{|x|^{m-1}}$$

for each $x \in \mathbb{R}^m - \mathscr{U}(0;1/\delta_0)$ we have

$$\int_{\mathbb{R}^m} |\operatorname{grad}\varphi_\delta|^2 \,\mathrm{d}\mathscr{H}_m \leqslant \int_{\mathbb{R}^m} |\operatorname{grad} R_\delta \mathscr{U}\nu_1|^2 \,\mathrm{d}\mathscr{H}_m + A\delta_0^{m-2}\sup|\psi'|L^2(2+\sup|\psi'|)$$

and (11) holds according to Lemma 6.

According to [28], Chapter V, §2, Theorem 1 there are $f_1,\dots,f_m \in L_2(\mathbb{R}^m)$ and a sequence $\delta_n \searrow 0$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^m}\Bigl(\frac{\partial}{\partial x_k}\varphi_{\delta_n}\Bigr) g \,\mathrm{d}\mathscr{H}_m = \int_{\mathbb{R}^m} f_k g \,\mathrm{d}\mathscr{H}_m \tag{12}$$

holds for each $g \in L_2(\mathbb{R}^m)$ and $k = 1,\dots,m$. Since Lemma 6 yields $\frac{\partial}{\partial x_k}\mathscr{U}\nu_2 \in L_2(\mathbb{R}^m)$ we obtain from (10) and (12)

$$\int_{\partial G}\mathscr{U}_c\nu_1 \,\mathrm{d}N^G\mathscr{U}\nu_2 = \int_G \sum_{k=1}^m f_k\Bigl(\frac{\partial}{\partial x_k}\mathscr{U}\nu_2\Bigr)\,\mathrm{d}\mathscr{H}_m.$$

It suffices to prove that $f_k = \frac{\partial}{\partial x_k}\mathscr{U}\nu_1$. Let $g \in L_2(\mathbb{R}^m)$ have a compact support disjoint with ∂G. Then

$$\int_{\mathbb{R}^m} f_k g \,\mathrm{d}\mathscr{H}_m = \lim_{n\to\infty}\int_{\mathbb{R}^m} g\frac{\partial}{\partial x_k}\varphi_{\delta_n}\,\mathrm{d}\mathscr{H}_m = \int_{\mathbb{R}^m} g\frac{\partial}{\partial x_k}\mathscr{U}\nu_1 \,\mathrm{d}\mathscr{H}_m$$

by Lemma 5. Since $\mathscr{H}_m(\partial G) = 0$, the set of such g is dense in $L_2(\mathbb{R}^m)$. Since $\frac{\partial}{\partial x_k}\mathscr{U}\nu_1 \in L_2(\mathbb{R}^m)$ by Lemma 6, we have $f_k = \frac{\partial}{\partial x_k}\mathscr{U}\nu_1$. $\square$

Lemma 8. *If G is bounded then there is a positive $\nu \in \mathscr{C}'$ such that $(T^G)^*\nu = -\nu$ and $\mathscr{U}\nu$ is constant in G.*

P r o o f. According to [11], Chapter II, §1 and §4 there is a positive measure ν on $\operatorname{cl} G$, a constant L and a Borel set K of null capacity such that $\mathscr{U}\nu = L$ on $\operatorname{cl} G - K$. Since $\mathscr{H}_{m-1}(K) = 0$ by [11], Theorem 3.13 and $\mathscr{U}\nu$ is lower semicontinuous by [11], Theorem 1.3, we obtain $\mathscr{U}\nu \leqslant L$ in G. Since $\mathscr{U}\nu$ is super-mean-valued by [11], Theorem 1.4 we have $\mathscr{U}\nu = L$ in G. Since $\Delta\mathscr{U}\nu = -\nu$ in the sense of distributions (see [9], Remark 5.7) and $\Delta\mathscr{U}\nu = 0$ in G obviously ν is supported by ∂G. If $\varphi \in \mathscr{D}$ then $\langle\varphi, N^G\mathscr{U}\nu\rangle = \int_G \operatorname{grad}\varphi\cdot\operatorname{grad}\mathscr{U}\nu \,\mathrm{d}\mathscr{H}_m = 0$ and thus $\bigl[(T^G)^* + I\bigr]\nu = \frac{1}{2}N^G\mathscr{U}\nu = 0$. $\square$

Lemma 9. *If $\nu \in \mathscr{C}'$, $\nu(\mathbb{R}^m) = 0$ then $(N^G \mathscr{U} \nu)(\mathbb{R}^m) = 0$.*

P r o o f. If G is bounded, choose $\varphi \in \mathscr{D}$, $\varphi \equiv 1$ on a neighbourhood of $\operatorname{cl} G$. Then

$$(N^G \mathscr{U} \nu)(\mathbb{R}^m) = \langle \varphi, N^G \mathscr{U} \nu \rangle = \int_G \operatorname{grad} \varphi \cdot \operatorname{grad} \mathscr{U} \nu = 0.$$

If G is unbounded then C is bounded. Since

$$N^G \mathscr{U} \nu = \frac{1}{2}[I + (T^G)^*]\nu = \frac{1}{2}[I - (T^C)^*]\nu = \frac{1}{2}(2I - N^C \mathscr{U})\nu$$

we have

$$(N^G \mathscr{U} \nu)(\mathbb{R}^m) = \nu(\mathbb{R}^m) - \frac{1}{2}(N^C \mathscr{U} \nu)(\mathbb{R}^m) = 0.$$

$\square$

Lemma 10. *Let λ_1, λ_2 be complex numbers, $\nu_1, \nu_2 \in {}^\wedge\mathscr{C}'$, $\nu_i(\mathbb{R}^m) \neq 0$, $N^G \mathscr{U} \nu_i = \lambda_i \nu_i$ for $i = 1, 2$. Then $\lambda_1 = \lambda_2$.*

P r o o f. Put ${}^\wedge\mathscr{C}'_0 = \{\mu \in {}^\wedge\mathscr{C}'; \mu(\mathbb{R}^m) = 0\}$. Then there are $\mu \in {}^\wedge\mathscr{C}'_0$ and a complex number α such that

$$\nu_2 = \alpha \nu_1 + \mu.$$

Then

$$\lambda_1 \alpha \nu_1 + N^G \mathscr{U} \mu = N^G \mathscr{U}(\alpha \nu_1 + \mu) = N^G \mathscr{U} \nu_2 = \lambda_2 \nu_2 = \lambda_2 \alpha \nu_1 + \lambda_2 \mu.$$

Hence

$$(\lambda_1 - \lambda_2)\alpha \nu_1 = \lambda_2 \mu - N^G \mathscr{U} \mu.$$

Since $\lambda_2 \mu - N^G \mathscr{U} \mu \in {}^\wedge\mathscr{C}'_0$ by Lemma 9, necessarily $(\lambda_1 - \lambda_2) = 0$. $\square$

Proposition 1. *Suppose $r_{\text{ess}}(T^G) < 1$. Let λ be an eigenvalue of $(T^G)^*$, $|\lambda| \geqslant 1$. Then $\lambda \in \{-1; 1\}$.*

P r o o f. Choose $\nu \in {}^\wedge\mathscr{C}'$, an eigenvector corresponding to the eigenvalue λ. Since $(T^G)^* = -(T^C)^*$ Lemma 8 yields that there is a positive measure $\mu \in \mathscr{C}'$ such that $(T^G)^* \mu = -\mu$ for G bounded and $(T^G)^* \mu = \mu$ for C bounded. If $\nu(\mathbb{R}^m) \neq 0$ then $\lambda \in \{-1; 1\}$ by Lemma 10.

Suppose $\nu(\mathbb{R}^m) = 0$. Denote by $\bar{\nu}$ the complex conjugate of ν. Since $\nu \in {}^\wedge\mathscr{C}'_c$ by Lemma 4 we obtain from Lemma 2 and Lemma 7

$$\begin{aligned}\int_G |\operatorname{grad} \mathscr{U} \nu|^2 &= \int_{\partial G} \mathscr{U}_c \bar{\nu} \, \mathrm{d} N^G \mathscr{U} \nu = \frac{1}{2} \int_{\partial G} \mathscr{U}_c \bar{\nu} \, \mathrm{d}(T^G + I)^* \nu = \frac{\lambda + 1}{2} \int_{\partial G} \mathscr{U}_c \bar{\nu} \, \mathrm{d}\nu \\ &= \frac{\lambda + 1}{2} \int_{\partial G} \mathscr{U}_c \bar{\nu} \, \mathrm{d}(N^G \mathscr{U} \nu + N^C \mathscr{U} \nu) = \frac{\lambda + 1}{2} \int_{\mathbb{R}^m} |\operatorname{grad} \mathscr{U} \nu|^2\end{aligned}$$

If

$$\int_{\mathbb{R}^m} |\operatorname{grad} \mathscr{U}\nu|^2 \neq 0$$

then $0 \leqslant \frac{1}{2}(\lambda+1) \leqslant 1$ and $\lambda \in \{-1;1\}$ because $|\lambda| \geqslant 1$. If

$$\int_{\mathbb{R}^m} |\operatorname{grad} \mathscr{U}\nu|^2 = 0$$

then $\mathscr{U}\nu$ is constant on G and on C. Since $\mathscr{U}_c\nu$ is continuous and

$$\lim_{|x|\to\infty} |\mathscr{U}\nu(x)| = 0$$

we have $\mathscr{U}_c\nu \equiv 0$. Since $\mathscr{H}_m(\partial G) = 0$ by Lemma 2 we obtain $\nu = 0$ by [11], Theorem 1.12 and Theorem 1.12', which is a contradiction. $\square$

Lemma 11. *Let $\nu \in {}^\wedge\mathscr{C}'$, $\nu(\mathbb{R}^m) \neq 0$, $(T^G)^*\nu = \lambda\nu$, $\lambda \neq 0$. Then there is no $\mu \in {}^\wedge\mathscr{C}'$ such that $\bigl[\lambda I - (T^G)^*\bigr]\mu = \nu$.*

P r o o f. Suppose that there is such a $\mu \in {}^\wedge\mathscr{C}'$. Then there are a complex number α and $\mu' \in {}^\wedge\mathscr{C}'_0 = \{\varrho \in {}^\wedge\mathscr{C}';\ \varrho(R^m) = 0\}$ such that $\mu = \alpha\nu + \mu'$. Then $\nu = \bigl[\lambda I - (T^G)^*\bigr]\mu = \bigl[\lambda I - (T^G)^*\bigr]\mu' \in {}^\wedge\mathscr{C}'_0$ by Lemma 9, which is a contradiction. $\square$

Proposition 2. *Suppose $r_{\mathrm{ess}}(T^G) < 1$. Let λ be an eigenvalue of the operator $(T^G)^*$, let $\nu \in {}^\wedge\mathscr{C}'$ be a corresponding eigenvector. If $|\lambda| \geqslant 1$ then there is no $\mu \in {}^\wedge\mathscr{C}'$ such that*

$$\bigl[\lambda I - (T^G)^*\bigr]\mu = \nu.$$

P r o o f. According to Lemma 11 it suffices to suppose $\nu(\mathbb{R}^m) = 0$. Suppose that there exists such a μ. According to Proposition 1 we have

$$N^G\mathscr{U}\nu = 0, \quad N^G\mathscr{U}\mu = -\frac{1}{2}\nu, \tag{13}$$

or

$$N^C\mathscr{U}\nu = 0, \quad N^C\mathscr{U}\mu = \frac{1}{2}\nu.$$

We can suppose that $\mu \in \mathscr{C}'$, $\nu \in \mathscr{C}'$. Lemma 4 yields that $\mu \in \mathscr{C}'_c$, $\nu \in \mathscr{C}'_c$. If (13) holds we obtain by Lemma 7 and Lemma 2

$$\begin{aligned} 0 &= \int_{\partial G} \mathscr{U}_c\mu \,\mathrm{d}N^G\mathscr{U}\nu - \int_{\partial G} \mathscr{U}_c\nu \,\mathrm{d}N^G\mathscr{U}\mu = \frac{1}{2}\int_{\partial G} \mathscr{U}_c\nu \,\mathrm{d}\nu \\ &= \frac{1}{2}\int_{\partial G} \mathscr{U}_c\nu \,\mathrm{d}\bigl[N^G\mathscr{U}\nu + N^C\nu\bigr] = \frac{1}{2}\int_{\mathbb{R}^m} |\operatorname{grad}\mathscr{U}\nu|^2 \,\mathrm{d}\mathscr{H}_m. \end{aligned}$$

Since $\lim\limits_{|x|\to\infty} |\mathscr{U}\nu(x)| = 0$ we have $\mathscr{U}_c\nu \equiv 0$. Since $\mathscr{H}_m(\partial G) = 0$ we have $\nu = 0$ by [11], Theorem 1.12 and Theorem 1.12$'$, which is a contradiction. The other case is analogical. $\square$

Proposition 3. *Let X be a complex Banach space and T a bounded linear operator on X. Suppose that $\lambda_1, \ldots, \lambda_k$ are different complex numbers such that* $\min\{|\lambda_1|, \ldots, |\lambda_k|\} > r > r_{\mathrm{ess}}(T)$. *Suppose that $\sigma(T) \cap \{\lambda; |\lambda| > r\} \subset \{\lambda_1, \ldots, \lambda_k\}$ and* $\operatorname{Ker}(\lambda_j I - T) = \operatorname{Ker}\big((\lambda_j I - T)^2\big)$ *for $j = 1, \ldots, k$, where $\sigma(T)$ denotes the spectrum of the operator T and* $\operatorname{Ker}(\lambda_j I - T)$ *is the null space of the operator $(\lambda_j I - T)$. Denote*

$$P(\lambda) = \prod_{j=2}^{k}(\lambda - \lambda_j) \qquad \text{for } k > 1,$$

$$1 \qquad \text{for } k = 1,$$

$$Q(\lambda) = \frac{P(\lambda) - P(\lambda_1)}{\lambda - \lambda_1}.$$

Then there are constants $M > 0$, $q \in (0; 1)$ such that for each $y \in (\lambda_1 I - T)(X)$ and any natural number n we have

$$\|(\lambda_1^{-1}T)^n P(T)y\| \leqslant M q^n \|y\| \tag{14}$$

and the series

$$P(\lambda_1)^{-1}\Big[Q(T)y + \lambda_1^{-1}\sum_{j=0}^{\infty}(\lambda_1^{-1}T)^j P(T)y\Big] \tag{15}$$

is a solution of the equation

$$(\lambda_1 I - T)x = y. \tag{16}$$

P r o o f. Put $\sigma_j = \sigma(T) \cap \{\lambda_j\}$ for $j = 1, \ldots, k$. Put $\sigma_{k+1} = \sigma(T) - \{\lambda_1, \ldots, \lambda_k\}$. Let P_j be the spectral projection corresponding to the spectral set σ_j for $j = 1, \ldots, k+1$ (see [26], Chapter VI, §4). Then $P_1 + \ldots + P_{k+1} = I$ and X is a direct sum of the spaces $P_1(X), \ldots, P_{k+1}(X)$.

Since T maps $P_{k+1}(X)$ into $P_{k+1}(X)$ and the restriction of T on $P_{k+1}(X)$ has a spectral radius smaller then or equal to r there are constants $K > 0$ and $q \in (0, 1)$ such that

$$\|(\lambda_1^{-1}T)^n y\| \leqslant K q^n \|y\| \tag{17}$$

for each $y \in P_{k+1}(X)$.

Fix $j \in \{1, \ldots, k\}$. If $\sigma_j = \emptyset$ then $P_j = 0$ and $P_j(X) = \{0\} = \operatorname{Ker}(\lambda_j I - T)$, $\operatorname{Ker} P_j = (\lambda_j I - T)(X)$. Now, let $\sigma_j = \{\lambda_j\}$. Since $r_{\mathrm{ess}}(T) < |\lambda_j|$ the operator

$(\lambda_j I - T)$ is Fredholm with index 0 by [26], Chapter VII, Theorem 5.4. According to [26], Chapter V, Theorem 2.3 the operator $(\lambda_j I - A)^2$ is Fredholm with index 0,too. Since $\operatorname{codim}(\lambda_j I - T)(X) = \dim \operatorname{Ker}(\lambda_j I - T) = \dim \operatorname{Ker}(\lambda_j I - T)^2 = \operatorname{codim}(\lambda_j I - T)^2(X)$ and $(\lambda_j I - T)^2(X) \subset (\lambda_j I - T)(X)$ we have $(\lambda_j I - T)^2(X) = (\lambda_j I - T)(X)$. By [8], Satz 50.2 we have $P_j(X) = \operatorname{Ker}(\lambda_j I - T)$, $\operatorname{Ker} P_j = (\lambda_j I - T)(X)$.

Now let $y \in (\lambda_1 I - T)(X)$. Since $(\lambda_1 I - T)(X) = \operatorname{Ker} P_1$ we have

$$y = \sum_{j=2}^{k+1} P_j y.$$

Since $P_j(X) = \operatorname{Ker}(\lambda_j I - T)$ for $j = 2, \ldots, k$ and thus $P(T)P_j y = 0$. We obtain

$$\|(\lambda_1^{-1}T)^n P(T)y\| = \|(\lambda_1^{-1}T)^n P(T)P_{k+1}y\| \leqslant Kq^n(\|P(T)\| \, \|P_{k+1}\| \, \|y\|,$$

because $P(T)P_{k+1}(X) \subset P_{k+1}(X)$. The series (15) converges and

$$\begin{aligned}&(\lambda_1 I - T)P(\lambda_1)^{-1}\big[Q(T)y + \lambda_1^{-1}\sum_{n=0}^{\infty}(\lambda_1^{-1}T)^n P(T)y\big]\\ &= P(\lambda_1)^{-1}\big[P(\lambda_1)y - P(T)y + \sum_{n=0}^{\infty}(\lambda_1^{-1}T)^n P(T)y - \sum_{n=1}^{\infty}(\lambda_1^{-1}T)^n P(T)y\big] = y.\end{aligned}$$

□

Lemma 12. *Suppose $r_{\mathrm{ess}}(T^G) < 1$. Denote by $H_1, \ldots, H_p$ the components of $\operatorname{cl} G$. Suppose that $\nu \in \mathscr{C}'$ satisfies $N^G \mathscr{U}\nu = 0$. Then there are $c_1, \ldots, c_p \in \mathbb{R}^1$ such that $\mathscr{U}\nu = c_i$ on int H_i.*

Proof. Suppose that $\nu(\mathbb{R}^m) = 0$. Since $\nu \in \mathscr{C}'_c$ by Lemma 4 we obtain from Lemma 7

$$0 = \int_{\partial G} \mathscr{U}_c \nu \, \mathrm{d}N^G \mathscr{U}\nu = \int_G |\operatorname{grad} \mathscr{U}\nu|^2 \, \mathrm{d}\mathscr{H}_m.$$

Therefore $\mathscr{U}\nu$ is constant on each component of G. Since $\mathscr{U}_c\nu$ is continuous and $\mathscr{U}\nu = \mathscr{U}_c\nu$ on $\mathbb{R}^m - \partial G$, $\mathscr{U}\nu$ is constant on int H_i.

Suppose now that $\nu(\mathbb{R}^m) \neq 0$. If G is bounded, Lemma 8 yields that there is $\lambda \in \mathscr{C}'$ such that $N^G \mathscr{U}\lambda = 0$, $\lambda(\mathbb{R}^m) \neq 0$ and $\mathscr{U}\lambda$ is constant on G. Thus

$$\mathscr{U}\nu = \frac{\nu(\mathbb{R}^m)}{\lambda(\mathbb{R}^m)}\mathscr{U}\lambda + \mathscr{U}\left(\nu - \frac{\nu(\mathbb{R}^m)}{\lambda(\mathbb{R}^m)}\lambda\right)$$

is constant on int H_i.

If G is not bounded, Lemma 8 yields that there is $\lambda \in \mathscr{C}'$, $\lambda(\mathbb{R}^m) \neq 0$ such that

$$T^G\lambda = -T^C\lambda = \lambda,$$

which is a contradiction with Lemma 10. □

Theorem 1. *Suppose that $r_{\mathrm{ess}}(T^G) < 1$. If $\mu \in \mathscr{C}'$ then the Neumann problem with the boundary condition μ has a solution if and only if $\mu \in \mathscr{C}'_0$ (= the space of such $\nu \in \mathscr{C}'$ for which $\nu(\partial H) = 0$ for each bounded component H of $\operatorname{cl} G$). We can take a solution in the form of the single layer potential $\mathscr{U}\nu$ where*

$$\nu = \mu + \sum_{j=0}^{\infty} \bigl[(-T^G)^*\bigr]^j \bigl[I - (T^G)^*\bigr]\mu. \tag{18}$$

Moreover, there are constants $M > 0$, $q \in (0;1)$ such that

$$\|\bigl[(-T^G)^*\bigr]^j \bigl[I - (T^G)^*\bigr]\mu\| \leqslant M q^j \|\mu\| \tag{19}$$

for each $\mu \in \mathscr{C}'_0$ and any natural number j.

If $\mathbb{R}^m - G$ is unbounded and connected then

$$\|\bigl[(-T^G)^*\bigr]^j \mu\| \leqslant M q^j \|\mu\| \tag{20}$$

for each $\mu \in \mathscr{C}'_0$ and any natural number j and

$$\nu = \sum_{j=0}^{\infty} \bigl[(-T^G)^*\bigr]^j 2\mu. \tag{21}$$

The series (21) converges for each $\mu \in \mathscr{C}'_0$ if and only if $\mathbb{R}^m - G$ is unbounded and connected.

Proof. Let $\mu \in \mathscr{C}'$, h be a solution of the Neumann problem with the boundary condition μ. Let H be a bounded component of $\operatorname{cl} G$. Since $\operatorname{cl} G$ has a finite number of components by Lemma 3, we can choose $\varphi \in \mathscr{D}$ such that $\varphi = 1$ on H and $\varphi = 0$ on $\operatorname{cl} G - H$. Then

$$\mu(\partial H) = \langle \varphi, \mu \rangle = \int_G \operatorname{grad} h \cdot \operatorname{grad} \varphi = 0.$$

Let $H_1, \ldots, H_p$ be all bounded components of $\operatorname{cl} G$. We are going to prove that

$$N^G \mathscr{U}(\mathscr{C}') = \{\mu \in \mathscr{C}';\ \mu(\partial H_i) = 0; i = 1, \ldots, p\}.$$

Since $\mathscr{U}\nu$ is a solution of the Neumann problem with the boundary condition $N^G\mathscr{U}\nu$ we have

$$N^G \mathscr{U}(\mathscr{C}') \subset \{\mu \in \mathscr{C}';\ \mu(\partial H_i) = 0; i = 1, \ldots, p\}.$$

Since

$$p = \operatorname{codim}\{\mu \in \mathscr{C}'\,;\ \mu(\partial H_i) = 0; i = 1, \dots, p\} \leqslant \operatorname{codim} N^G \mathscr{U}(\mathscr{C}') = \dim \operatorname{Ker} N^G \mathscr{U}$$

because $N^G\mathscr{U}$ is a Fredholm operator with index 0, it suffices to prove that $\dim \operatorname{Ker} N^G \mathscr{U} \leqslant p$.

If $\nu \in \operatorname{Ker} N^G\mathscr{U}$ then $\nu \in \mathscr{C}'_c$ by Lemma 4 and $\mathscr{U}_c\nu$ remains constant on each component of $\operatorname{cl} G$ by Lemma 12. If G is unbounded and H_0 is the unbounded component of $\operatorname{cl} G$ then $\mathscr{U}_c\nu$ must vanish on H_0. This is clear provided $m > 2$, because then $\mathscr{U}\nu$ tends to zero at infinity, while for $m = 2$ the relation

$$\lim_{|x|\to\infty} \left| \mathscr{U}\nu(x) + \frac{1}{2\pi}\nu(\partial G)\log|x| \right| = 0$$

shows that the potential $\mathscr{U}\nu$ can remain constant on H_0 only if $\nu(\partial G) = 0$ when its limit at infinity equals zero.

If $\nu \in \mathscr{C}'_c$, $\mathscr{U}\nu = 0$ in G, $\mathscr{U}\nu$ converges to 0 at infinity then $\mathscr{U}_c\nu$ is a harmonic function in $\mathbb{R}^m - \partial G$ which vanishes on ∂G and converges to 0 at infinity, hence $\mathscr{U}\nu = \mathscr{U}_c\nu = 0$ in $\mathbb{R}^m - \partial G$. Since $\mathscr{H}_m(\partial G) = 0$ by Lemma 2, we obtain $\nu = 0$ by [11], Theorem 1.12, Theorem 1.12′.

If there is no $\mu \in \mathscr{C}'$ with $\mu(\partial G) \neq 0$ such that $\mathscr{U}\mu$ vanishes identically on G then $\dim \operatorname{Ker} N^G\mathscr{U} \leqslant p$. Suppose now that there exists such a μ. Then $m = 2$ and G is bounded. We are going to prove that there is no $\nu \in \mathscr{C}'$, $\nu(\partial G) = 0$ such that $\mathscr{U}\nu = 1$ on G. It yields that $\dim \operatorname{Ker} N^G\mathscr{U} \leqslant p$.

Fix $r > 1$ large enough to guarantee $\operatorname{cl} G \subset \mathscr{U}(0; r)$ and consider a probability measure $\mathscr{H}$ distributed on $\partial\mathscr{U}(0; r)$ with a constant density with respect to $\mathscr{H}_1$. As is noticed in [9], Remark 5.10,

$$\mathscr{U}\mathscr{H} = \frac{1}{2\pi}\log\frac{1}{r} \qquad \text{on } \mathscr{U}(0; r) \supset \operatorname{cl} G.$$

Fubini's theorem implies the reciprocity law

$$\int_{\mathbb{R}^2} \mathscr{U}\nu \,\mathrm{d}\mathscr{H} = \int_{\mathbb{R}^2} \mathscr{U}\mathscr{H} \,\mathrm{d}\nu. \tag{22}$$

Now $\mathscr{U}\nu$ (being harmonic on $\mathbb{R}^2 - \operatorname{cl} G$ and tending to 1 at $\partial(\mathbb{R}^2 - \operatorname{cl} G)$ and to zero at infinity) remains positive on $\mathbb{R}^2 - \operatorname{cl} G \supset \partial\mathscr{U}(0; r)$, so that the left-hand side of (22) is positive, while the right-hand side equals $\nu(\partial G)\frac{1}{2\pi}\log\frac{1}{r} = 0$. (Compare [9], proof of Proposition 5.11.)

We have proved that there is a solution of the Neumann problem with the boundary condition $\mu \in \mathscr{C}'$ if and only if $\mu \in \mathscr{C}'_0$ and we can take a solution in the form of the single layer potential $\mathscr{U}\nu$ where

$$\big[I + (T^G)^*\big]\nu = 2\mu.$$

Propositions 1, 2 and 3 yield the relations (18), (19), (20), (21).

Suppose now that $\mathbb{R}^m - G$ is not unbounded and connected. Since $\operatorname{cl} C$ has a bounded component and $r_{\mathrm{ess}}(T^C) = r_{\mathrm{ess}}(T^G)$ we have

$$\big[I - (T^G)^*\big](\mathscr{C}') = \big[I + (T^C)^*\big](\mathscr{C}') = N^C\mathscr{U}(\mathscr{C}') \subsetneqq \mathscr{C}'.$$

Since $I-(T^G)^*$ is a Fredholm operator with index 0 by [26], Chapter IX, Theorem 2.1, Theorem 1.3 and Chapter VII, Theorem 3.5, there is a $\mu \in \mathscr{C}'$, $\mu \neq 0$ such that $(T^G)^*\mu = \mu$. Since $\mu = \frac{1}{2}N^G\mathscr{U}\mu$ we have $\mu \in \mathscr{C}'_0$. But the series (21) diverges. $\square$

Example 1. Consider $G = \mathscr{U}(0;r) \subset \mathbb{R}^2$. For $f \in \mathscr{C}$, $x \in \partial G$ we can calculate

$$\begin{aligned} T^G f(x) &= -2\int_{\partial G} f(y)\frac{y}{r}\cdot\frac{1}{2\pi}\frac{y-x}{|x-y|^2}\,\mathrm{d}\mathscr{H}_1(y) \\ &= -\int_{\partial G} f(y)\frac{1}{2\pi r}\frac{|y|^2+|x|^2-2y\cdot x}{|x-y|^2}\,\mathrm{d}\mathscr{H}_1(y) = -\frac{1}{2\pi r}\int_{\partial G} f(y)\,\mathrm{d}\mathscr{H}_1(y). \end{aligned}$$

Hence

$$(T^G)^*\mu = \mu(\partial G)\mathscr{H},$$

where

$$\int_{\partial G} f\,\mathrm{d}\mathscr{H} = -\frac{1}{2\pi r}\int_{\partial G} f\,\mathrm{d}\mathscr{H}_1(y).$$

Using Theorem 1 we obtain that for $\mu \in \mathscr{C}'$ for which $\mu(\partial G) = 0$ we can take a solution of the Neumann problem with the boundary condition μ in the form

$$\frac{1}{\pi}\int_{\partial\mathscr{U}(0;r)} \log\frac{1}{|x-y|}\,\mathrm{d}\mu(y).$$

Example 2. Consider $G = \mathbb{R}^2 - \mathscr{U}(0;r)$. Since $T^G = -T^C$ we obtain from Example 1 that

$$(T^G)^*\mu = \mu(\partial G)\mathscr{H},$$

where

$$\int_{\partial G} f\,\mathrm{d}\mathscr{H} = +\frac{1}{2\pi r}\int_{\partial G} f(y)\,\mathrm{d}\mathscr{H}_1(y).$$

Using Theorem 1 we obtain that for $\mu \in \mathscr{C}'$ we can take a solution of the Neumann problem with the boundary condition μ in the form

$$\frac{1}{\pi}\int_{\partial\mathscr{U}(0;r)} \log\frac{1}{|x-y|}\,\mathrm{d}\mu(y) - \frac{\mu(\mathbb{R}^m)}{4\pi^2 r}\int_{\partial\mathscr{U}(0;r)} \log\frac{1}{|x-y|}\,\mathrm{d}\mathscr{H}_1(y).$$

Since

$$\frac{1}{2\pi r}\int_{\partial\mathscr{U}(0;r)} \log\frac{1}{|x-y|}\,\mathrm{d}\mathscr{H}_1(y) - \log\frac{1}{|x|}$$

is a harmonic function on G which vanishes on ∂G by [9], Remark 5.10 and tends to zero at infinity it vanishes in G. Thus

$$\frac{1}{\pi}\int_{\partial\mathscr{U}(0;r)} \log\frac{1}{|x-y|}\,\mathrm{d}\mu(y) - \frac{\mu(\mathbb{R}^m)}{2\pi}\log\frac{1}{|x|}$$

is a solution of the Neumann problem with the boundary condition μ.

Theorem 2. *Suppose that $r_{\mathrm{ess}}(T^G) < 1$ and $\operatorname{cl} G$ is unbounded and connected. Then there are constants $M > 0$, $q \in (0;1)$ such that*

$$\|(-T^G)^j(I - T^G)f\| \leqslant Mq^j\|f\| \tag{23}$$

for each $f \in \mathscr{C}$ and any natural number j. The solution of the Dirichlet problem for C with the boundary condition $g \in \mathscr{C}$ is the double layer potential

$$W^G f(x) = \frac{1}{A}\int_{\partial G} f(y) n^G(y)\cdot\frac{y-x}{|y-x|^m}\,\mathrm{d}\mathscr{H}_{m-1}(y),$$

where

$$f = g + \sum_{j=0}^{\infty}(-T^G)^j(I - T^G)g. \tag{24}$$

Proof. Since $\lambda I + T^G$ is a Fredholm operator with index 0 for $|\lambda| \geqslant 1$, we have $\sigma(T^G) \cap \{\lambda; |\lambda| \geqslant 1\} \subset \{-1;1\}$ by Proposition 1, [28], Chapter VIII, §6, Lemma 1 and [26], Chapter VII, Theorem 3.5. Since there is a natural number n and a linear compact operator K on ${}^{\wedge}\mathscr{C}$ such that $\|(T^G)^n + K\| < 1$ we obtain from [13], Lemma 2 that $\sigma\big((T^G)^n\big) \cap \{\lambda; |\lambda| \geqslant 1\}$ is an isolated subset of $\sigma\big((T^G)^n\big)$. Since $\sigma\big((T^G)^n\big) = \{\lambda^n;\ \lambda \in \sigma(T^G)\}$ by [28], Chapter VIII, §7, the set $\sigma(T^G) \cap \{\lambda;\ |\lambda| \geqslant 1\}$ is an isolated subset of $\sigma(T^G)$. Theorem 1 yields that $(I + T^G)^*(\mathscr{C}') = \mathscr{C}'$. Since $(I + T^G)$ is a Fredholm operator of index 0 we have $\operatorname{Ker}\big((I + T^G)^*\big) = \{0\}$. Since $I + T^G$ is a Fredholm operator we have $(I + T^G)(\mathscr{C}) = \mathscr{C}$ by [28], Chapter VII, §5. Now, the assertion of the theorem is a consequence of Proposition 3. $\square$

Note 3. Suppose that $r_{\mathrm{ess}}(T^G) < 1$, $\mathrm{cl}\, G$ is unbounded and connected, $g \in \mathscr{C}$. Let M, q be the constants from Theorem 2. Since

$$\sup_{x \in C} |W^G h(x)| \leqslant \|h\| \Big(V^G + \frac{1}{2} \Big)$$

for each $h \in \mathscr{C}$ by [9], Theorem 2.16, we obtain from Theorem 2

$$\sup_{x \in C} |W^G g_j(x)| \leqslant M(V^G + \frac{1}{2}) q^j \|g\|$$

where

$$g_j = (-T^G)^j (I - T^G) g.$$

So, the series

$$W^G g + \sum_{j=0}^{\infty} W^G g_j$$

converges absolutely uniformly on C to $W^G f$, the solution of the Dirichlet problem for C with the boundary condition g, where f is given by (24). Besides,

$$\sup_{x \in C} |W^G f| \leqslant (V^G + 1) \Big(1 + \|T^G\| + 1 + \sum_{j=1}^{\infty} M q^j \Big) \|g\|.$$

Note 4. Fix $x_0 \in \partial \mathscr{U}(0;1)$. Then $-\frac{1}{\pi} \lg |x - x_0|$ is a solution of the Neumann problem for $\mathscr{U}(0;1)$ with the boundary condition δ_{x_0} (= the Dirac measure supported in $\{x_0\}$). But the function $-\frac{1}{\pi} \lg |x - x_0|$ is not bounded in $\mathscr{U}(0;1)$. So, for the Neumann problem we cannot obtain the same estimates as for the Dirichlet problem in Note 3. Nevertheless, if $r_{\mathrm{ess}}(T^G) < 1$ then there exists $q \in (0;1)$ such that for each compact $K \subset G$ there is a constant M_K such that

$$\sup_{x \in K} |\mathscr{U} \mu(x)| \leqslant M_K \|\mu\|,$$
$$\sup_{x \in K} |\mathscr{U} \mu_j(x)| \leqslant M_K q^j \|\mu\|$$

for each $\mu \in \mathscr{C}_0'$, where

$$\mu_j = \big[(-T^G)^* \big]^j \big[I - (T^G)^* \big] \mu$$

so that the series

$$\mathscr{U} \mu + \sum_{j=0}^{\infty} \mathscr{U} \mu_j$$

converges locally uniformly in G to the solution of the Neumann problem with the boundary condition μ and

$$\sup_{x\in K}\left|\mathscr{U}\mu(x)+\sum_{j=0}^{\infty}\mathscr{U}\mu_j(x)\right|\leqslant M_K\Big(1+\frac{1}{1-q}\Big)\|\mu\|.$$

Note 5. Denote by $\mathscr{H}$ the restriction of $\mathscr{H}_{m-1}$ to $\widehat{\partial}G$. Denote by $L_1(\mathscr{H})$ the space of all functions f measurable with respect to $\mathscr{H}$ such that

$$\int_{\partial G}|f|\,\mathrm{d}\mathscr{H}<\infty.$$

For $f\in L_1(\mathscr{H})$ denote by $\nu_f\in\mathscr{C}'$ the measure

$$\nu_f(M)=\int_M f\,\mathrm{d}\mathscr{H}.$$

If $f\in L_1(\mathscr{H})$ then

$$(T^G)^*\nu_f=\nu_g$$

where

$$g(x)=T'f(x)=\frac{2}{A}\int_{\partial G}n(x)\cdot\frac{x-y}{|y-x|^m}f(y)\,\mathrm{d}\mathscr{H}(y).$$

Suppose that $r_{ess}(T^G)<1$. If $f\in L_1(\mathscr{H})$ and $\nu_f\in\mathscr{C}_0'$ then

$$g=f+\sum_{j=0}^{\infty}(-T')^j(I-T')f$$

converges in $L_1(\mathscr{H})$ and $\mathscr{U}\nu_g$ is a solution of the Neumann problem with the boundary condition ν_f.

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