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CONVERGENCE ESTIMATE FOR SECOND ORDER CAUCHY PROBLEMS WITH A SMALL PARAMETER

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Abstract. We consider the second order initial value problem in a Hilbert space, which is a singular perturbation of a first order initial value problem. The difference of the solution and its singular limit is estimated in terms of the small parameter ε. The coefficients are commuting self-adjoint operators and the estimates hold also for the semilinear problem.

1. Introduction

We consider the initial value problem in a Hilbert space X

$$\varepsilon u_{tt} + Au_t + Bu + f(u) = 0,\ u(0) = u_{0\varepsilon},\ u_t(0) = u_{1\varepsilon} \tag{1}$$

for $\varepsilon > 0$, and its limit

$$Au_t + Bu + f(u) = 0,\ u(0) = u_{00}. \tag{2}$$

The operators A and B are commuting positive self-adjoint operators in X. This problem has been throughly investigated when $A = aI$ (see [2]). If A is not a multiple of identity, two papers have recently appeared treating the commutative case. In [1], the space X is a Banach space, $f = 0$, B is the generator of a strongly continuous cosine family and A is a bounded operator commuting with B. In [3], the space X is a Hilbert space and A and B are commuting (in general unbounded) positive self-adjoint operators. There it is shown that under mild conditions on f, $u_{0\varepsilon}$ and $u_{1\varepsilon}$, the solutions u_ε of (1) converge locally uniformly in t to the solution u_0 of (2). However, the convergence rate for $u_\varepsilon - u_0$ has not been established. In [1], this convergence rate was estimated.

It is our aim to estimate $u_\varepsilon - u_0$ and its derivative $u_\varepsilon' - u_0'$ under the assumptions similar to those of [3].

We list our assumptions.

(A1) The operators A and B are commuting uniformly positive self-adjoint operators in X such that

$$B \text{ is } A^2\text{-bounded.} \tag{3}$$

This means that BA^{-2} is a bounded operator. The assumption (3) is sufficiently general to allow the applications described in Cases 1 and 2 in [3]. The assumption (3) implies that the damping term A is "large". In the other extreme case when A is bounded, the estimates from [1] apply.

In the first three results we consider the linear case. We always consider *mild* solutions of (1) and (2) (see [3]); when the initial data are sufficiently regular, these solutions have additional regularity properties.

Theorem 1. *Assume* (A1) *and* $f = 0$. *There exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $t \geqslant 0$ *and* $\varepsilon \in (0, \varepsilon_0)$ *the following estimate holds:*

$$\begin{aligned} \|u_\varepsilon(t) - u_0(t)\| \leqslant C[&\varepsilon(\|u_{0\varepsilon}\| + t\|BA^{-1}\mathrm{e}^{-tBA^{-1}}u_{0\varepsilon}\| + \|A^{-1}u_{1\varepsilon}\|) \\ &+ \|\mathrm{e}^{-tBA^{-1}}(u_{0\varepsilon} - u_{00})\|]. \end{aligned} \tag{4}$$

The next estimates follow from (4).

Proposition 2. *Assume* (A1) *and* $f = 0$. *There exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $t \geqslant 0$ *and* $\varepsilon \in (0, \varepsilon_0)$ *the following estimates hold:*

$$\|u_\varepsilon(t) - u_0(t)\| \leqslant C[\varepsilon(\|u_{00}\| + \|A^{-1}u_{1\varepsilon}\|) + \|u_{0\varepsilon} - u_{00}\|], \tag{5}$$

$$\|u_\varepsilon'(t) - u_0'(t)\| \leqslant C[\varepsilon\|BA^{-1}u_{00}\| + \|u_{1\varepsilon} + BA^{-1}u_{0\varepsilon}\| + \|u_{1\varepsilon} + BA^{-1}u_{00}\|]. \tag{6}$$

Proposition 3. *Assume* $f = 0$. *If* A *is* B*-bounded and* $\gamma \leqslant \frac{1}{\|AB^{-1}\|}$, *then for every* $\delta \in (0, \gamma)$ *there exists* $C > 0$ *such that for all* $t \geqslant 0$ *and* $\varepsilon \in (0, \varepsilon_0)$ *the following estimate holds:*

$$\|u_\varepsilon(t) - u_0(t)\| \leqslant C[\varepsilon\left(\|u_{0\varepsilon}\| + \|A^{-1}u_{1\varepsilon}\|\right) + \mathrm{e}^{-(\gamma-\delta)t}\|u_{0\varepsilon} - u_{00}\|]. \tag{7}$$

The estimates (4)–(7) represent a strengthening of the results from [1] under the present assumptions.

Next we consider a general f. In addition to (A1) we assume

(A2)

$$B \text{ is } A\text{-bounded,} \tag{8}$$

(A3) the mapping $f\colon D(f) \to X$ is defined on $D(f) \supset D(B^{1/2})$ and

a) f is the Gateaux derivative of a positive convex functional F in X with the domain $D(F) \supset D(B^{1/2})$,

b) f is a locally Lipschitz mapping from $D(B^{1/2})$ into X in the following sense: for every $R > 0$ there exists $C > 0$ such that $\|A^{1/2}u\| \leqslant R$, $\|A^{1/2}v\| \leqslant R$ imply

$$\|f(u) - f(v)\| \leqslant C\|B^{1/2}(u - v)\|, \tag{9}$$

(A4) $u_{0\varepsilon} \in D(A)$ $(\varepsilon \geqslant 0)$, $u_{1\varepsilon} \in D(A^{1/2}) \cap D(B)$ $(\varepsilon > 0)$,

(A5) $\sup\limits_{\varepsilon}(\varepsilon^{1/2}\|u_{1\varepsilon}\| + \|A^{1/2}u_{0\varepsilon}\| + F(u_{0\varepsilon})) < \infty$,

(A6) $\lim\limits_{\varepsilon \to 0} B^{1/2}(u_{0\varepsilon} - u_{00}) = 0$.

It was shown in [3] that under the assumptions (A3)–(A6) the equation (1) has a global classical solution u_ε and the equation (2) has a global classical solution u_0. The following theorem will be proved.

Theorem 4. *Assume* (A2)–(A6). *Then there exist $C > 0$ and $\varepsilon_0 > 0$ such that for every $\beta \in [0, 1)$ there is $K_\beta > 0$ with the property that the estimate*

$$\|B^{1/2}[u_\varepsilon(t) - u_0(t)]\| \leqslant C\mathrm{e}^{K_\beta t}[\varepsilon(1 + t + \|u_{1\varepsilon}\|) + \varepsilon^\beta t^{1-\beta} + \|B^{1/2}(u_{0\varepsilon} - u_{00})\|] \tag{10}$$

holds.

2. The linear case

We first consider the initial value problems (1) and (2) with $f = 0$.

Since A and B are commuting self-adjoint operators, there exists a self-adjoint operator K in X and positive measurable functions a and b on $\sigma = \sigma(K)$ such that $A = a(K)$, $B = b(K)$. The assumption (A1) implies the existence of positive numbers μ and b_0 such that

$$0 < b_0 \leqslant b(\lambda) \leqslant \mu a(\lambda)^2 \quad (\lambda \in \sigma). \tag{11}$$

Setting

$$\varepsilon_0 = \frac{3}{16\mu} \tag{12}$$

and $q_\varepsilon(\lambda) = a(\lambda)^2 - 4\varepsilon b(\lambda)$, we conclude $q_\varepsilon(\lambda) \geqslant a(\lambda)^2 - 4\varepsilon\mu a(\lambda)^2 = (1 - \frac{3\varepsilon}{4\varepsilon_0})a(\lambda)^2 \geqslant \frac{1}{4}a(\lambda)^2$ if $\lambda \in \sigma$ and $\varepsilon \leqslant \varepsilon_0$, i.e.

$$\sqrt{q_\varepsilon(\lambda)} \geqslant \frac{1}{2}a(\lambda) \geqslant \sqrt{\frac{b_0}{4\mu}} \quad (0 \leqslant \varepsilon < \varepsilon_0,\ \lambda \in \sigma). \tag{13}$$

It was shown in [3] that if the condition (13) is satisfied, then the difference $u_\varepsilon(t) - u_0(t)$ can be represented as

$$u_\varepsilon(t) - u_0(t) = f_\varepsilon(t, K)u_{0\varepsilon} + s_\varepsilon(t, K)u_{1\varepsilon} + \mathrm{e}^{-tBA^{-1}}(u_{0\varepsilon} - u_{00}), \tag{14}$$

where

$$f_\varepsilon(t,\lambda) = \frac{a(\lambda)\mathrm{e}^{-\frac{ta(\lambda)}{2\varepsilon}} \operatorname{sh} \frac{t}{2\varepsilon}\sqrt{q_\varepsilon(\lambda)}}{\sqrt{q_\varepsilon(\lambda)}} + \mathrm{e}^{-\frac{ta(\lambda)}{2\varepsilon}} \operatorname{ch} \frac{t}{2\varepsilon}\sqrt{q_\varepsilon(\lambda)} - \mathrm{e}^{-\frac{tb(\lambda)}{a(\lambda)}},$$

$$s_\varepsilon(t,\lambda) = \frac{2\varepsilon\mathrm{e}^{-\frac{ta(\lambda)}{2\varepsilon}} \operatorname{sh} \frac{t}{2\varepsilon}\sqrt{q_\varepsilon(\lambda)}}{\sqrt{q_\varepsilon(\lambda)}}.$$

In order to estimate $u_\varepsilon - u_0$, we need a precise estimates of $\sup\limits_{\lambda\in\sigma}|f_\varepsilon(t,\lambda)|$, $\sup\limits_{\lambda\in\sigma}|s_\varepsilon(t,\lambda)|$. Note that the functions $(\varepsilon, t) \mapsto f_\varepsilon(t,\lambda), s_\varepsilon(t,\lambda)$ are not C^1 at $(0,0)$, which makes it hard to find uniform estimates without the condition (11). This is why we imposed the condition (3), which implies (13). The functions $f_\varepsilon(t,\lambda), s_\varepsilon(t,\lambda)$ are well behaved under the condition (13), and this will enable us to deduce the estimates (4)–(7).

We first prove the announced estimates of f_ε and s_ε.

Lemma 5. *There exists $C > 0$ such that the estimate*

$$|f_\varepsilon(t,\lambda)| \leqslant C\varepsilon\Big[1 + \frac{tb(\lambda)}{a(\lambda)}\Big]\mathrm{e}^{-\frac{tb(\lambda)}{a(\lambda)}} \tag{15}$$

holds for all $\lambda \in \sigma$, $t \geqslant 0$ and $\varepsilon \in (0, \varepsilon_0)$.

P r o o f. Denote $m_\varepsilon(\lambda) := a(\lambda) + \sqrt{q_\varepsilon(\lambda)}$; then $m_\varepsilon(\lambda) \geqslant \frac{3a(\lambda)}{2}$ by (13). Fix $\lambda \in \sigma$ and denote

$$b := b(\lambda), a := a(\lambda),\ q_\varepsilon := q_\varepsilon(\lambda),\ m_\varepsilon := m_\varepsilon(\lambda),\ f_\varepsilon(t) := f_\varepsilon(t,\lambda).$$

Note that

$$\mathrm{e}^{-\frac{ta}{2\varepsilon}}\mathrm{e}^{\frac{t}{2\varepsilon}\sqrt{q_\varepsilon}} = \mathrm{e}^{-\frac{2tb}{m_\varepsilon}},\ \mathrm{e}^{-\frac{ta}{2\varepsilon}}\mathrm{e}^{-\frac{t}{2\varepsilon}\sqrt{q_\varepsilon}} = \mathrm{e}^{-\frac{tm_\varepsilon}{2\varepsilon}},$$

which implies

$$\mathrm{e}^{-\frac{ta}{2\varepsilon}}\mathrm{e}^{\frac{t}{2\varepsilon}\sqrt{q_\varepsilon}} - \mathrm{e}^{-\frac{tb}{a}} = \mathrm{e}^{-\frac{tb}{a}}(\mathrm{e}^{-\frac{4\varepsilon tb^2}{am_\varepsilon^2}} - 1),$$

$$\mathrm{e}^{-\frac{ta}{2\varepsilon}}\mathrm{e}^{-\frac{t}{2\varepsilon}\sqrt{q_\varepsilon}} - \mathrm{e}^{-\frac{tb}{a}} = \mathrm{e}^{-\frac{tb}{a}}(\mathrm{e}^{-\frac{tm_\varepsilon^2}{4a\varepsilon}} - 1).$$

Define

$$g(\varepsilon,t) = 2\mathrm{e}^{\frac{tb}{a}}f_\varepsilon(t).$$

Note that $g(\cdot,t) \in C^1[0,\varepsilon_0)$ for all $t \geqslant 0$ (because of (13)) and that g can be represented as

$$g(\varepsilon,t) = \frac{m_\varepsilon}{\sqrt{q_\varepsilon}}\mathrm{e}^{-\frac{4\varepsilon tb^2}{am_\varepsilon^2}} - \frac{4\varepsilon b}{m_\varepsilon\sqrt{q_\varepsilon}}\mathrm{e}^{-\frac{tm_\varepsilon^2}{4a\varepsilon}} - 2. \tag{16}$$

Differentiating, we find

$$\begin{aligned} -\frac{\partial g(\varepsilon,t)}{\partial\varepsilon} &= \frac{2b}{q_\varepsilon}\mathrm{e}^{-\frac{4\varepsilon tb^2}{am_\varepsilon^2}}\Big(-\frac{a}{\sqrt{q_\varepsilon}} + \frac{2tbm_\varepsilon\sqrt{q_\varepsilon} + 8tb^2\varepsilon}{am_\varepsilon^2}\Big) \\ &\quad + \frac{b}{q_\varepsilon}\mathrm{e}^{-\frac{tm_\varepsilon^2}{4a\varepsilon}}\Big(\frac{8b\varepsilon}{m_\varepsilon\sqrt{q_\varepsilon}} + \frac{8b\varepsilon}{m_\varepsilon^2} + \frac{4\sqrt{q_\varepsilon}}{m_\varepsilon} + \frac{4bt}{a} + \frac{tm_\varepsilon\sqrt{q_\varepsilon}}{a\varepsilon}\Big). \end{aligned}$$

Using (13), this implies

$$\begin{aligned} \Big|\frac{\partial g(\varepsilon,t)}{\partial\varepsilon}\Big| &\leqslant 16\mu\frac{4\varepsilon tb^2}{am_\varepsilon^2}\mathrm{e}^{-\frac{4\varepsilon tb^2}{am_\varepsilon^2}} + 16\mu + \frac{256\mu^2}{9}\frac{bt}{a} \\ &\quad + \Big(\frac{16\mu}{3} + \frac{256\mu^2}{9}\varepsilon\Big)\frac{tm_\varepsilon^2}{4a\varepsilon}\mathrm{e}^{-\frac{tm_\varepsilon^2}{4a\varepsilon}} + \frac{512}{9}\mu^2\varepsilon + \frac{16\mu}{3}. \end{aligned}$$

Denote $r(x) = x\mathrm{e}^{-x}$. It follows that

$$\Big|\frac{\partial g(\varepsilon,t)}{\partial\varepsilon}\Big| \leqslant C\Big[r\Big(\frac{4\varepsilon tb^2}{am_\varepsilon^2}\Big) + 1 + \frac{bt}{a} + r\Big(\frac{tm_\varepsilon^2}{4a\varepsilon}\Big)\Big]. \tag{17}$$

Note that $r(0) = 0$ and $r(x) \leqslant 1$ $(x > 0)$. From $g(0,t) = 0$ $(t \geqslant 0)$ it follows that

$$|g(\varepsilon,t)| \leqslant \varepsilon\max_{0<\delta<\varepsilon}\Big|\frac{\partial g}{\partial\varepsilon}(\delta,t)\Big|.$$

Hence

$$|f_\varepsilon(t)| = \frac{1}{2}\mathrm{e}^{-\frac{tb}{a}}|g(\varepsilon,t)| \leqslant C\varepsilon\Big[(1+\tilde r_{a,b}(\varepsilon,t))\mathrm{e}^{-\frac{tb}{a}} + r\Big(\frac{bt}{a}\Big)\Big]$$

with C independent of a and b, $\tilde r_{a,b}(\varepsilon,0)=0$ and $0\leqslant \tilde r_{a,b}(\varepsilon,t)\leqslant 1$. This implies (15).

Lemma 6. *Let $\beta\geqslant 0$. There exists $C>0$ such that the estimate*

$$\Big|\frac{1}{\varepsilon}s_\varepsilon(t,\lambda) - \frac{1}{a(\lambda)}\mathrm{e}^{-\frac{tb(\lambda)}{a(\lambda)}}\Big| \leqslant C\Big[\frac{\varepsilon}{a(\lambda)} + \frac{\varepsilon^\beta}{a(\lambda)^{1+\beta}t^\beta}\Big] \tag{18}$$

holds for all $\lambda\in\sigma$, $t\geqslant 0$ and $\varepsilon\in(0,\varepsilon_0)$.

Proof. Fix λ and recall the notation a, b, $q_\varepsilon, m_\varepsilon$ from Lemma 5. Further denote $s_\varepsilon(t) := s_\varepsilon(t,\lambda)$ and

$$k(\varepsilon,t) = \frac{\sqrt{q_\varepsilon}}{a}\mathrm{e}^{\frac{tb}{a}}\Big[\frac{a}{\varepsilon}s_\varepsilon(t) - \frac{a}{\sqrt{q_\varepsilon}}\mathrm{e}^{-\frac{tb}{a}}\Big].$$

Then $k = k_1+k_2$ with

$$k_1(\varepsilon,t) = \mathrm{e}^{-\frac{4tb^2\varepsilon}{am_\varepsilon^2}} - 1,\ \ k_2(\varepsilon,t) = -\mathrm{e}^{-\frac{tm_\varepsilon^2}{4a\varepsilon}}.$$

From

$$-\frac{\partial k_1(\varepsilon,t)}{\partial\varepsilon} = \frac{4tb^2}{am_\varepsilon^3\sqrt{q_\varepsilon}}(m_\varepsilon\sqrt{q_\varepsilon}+4b\varepsilon)\mathrm{e}^{-\frac{4tb^2\varepsilon}{am_\varepsilon^2}}$$

it follows that $|\frac{\partial k_1}{\partial\varepsilon}(\varepsilon,t)| \leqslant \frac{4tb^2}{am_\varepsilon^2} + \frac{4b}{m_\varepsilon}\sqrt{q_\varepsilon}r(\frac{4tb^2\varepsilon}{am_\varepsilon^2})$. From (13) it follows that $|\frac{\partial k_1(\varepsilon,t)}{\partial\varepsilon}| \leqslant C(\frac{tb}{a}+1)$. Since $k_1(0,t)=0$ for all $t\geqslant 0$, we conclude $\frac{a}{\sqrt{q_\varepsilon}}\mathrm{e}^{-\frac{tb}{a}}|k_1(\varepsilon,t)|\leqslant C\varepsilon$.

Further, note that $x^\beta\mathrm{e}^{-x}\leqslant C_\beta$ $(x\geqslant 0)$, hence $(\frac{tm_\varepsilon^2}{4a\varepsilon})^\beta|k_2(\varepsilon,t)|\leqslant C_\beta$, implying $\frac{a}{\sqrt{q_\varepsilon}}\mathrm{e}^{-\frac{tb}{a}}|k_2(\varepsilon,t)|\leqslant C_\beta(\frac{\varepsilon}{at})^\beta$. This yields

$$\begin{aligned}\Big|\frac{1}{\varepsilon}s_\varepsilon(t) - \frac{1}{a}\mathrm{e}^{-\frac{tb}{a}}\Big| &= \Big|\frac{1}{a}\Big[\frac{a}{\varepsilon}s_\varepsilon(t) - \frac{a}{\sqrt{q_\varepsilon}}\mathrm{e}^{-\frac{tb}{a}}\Big] + \Big(\frac{1}{\sqrt{q_\varepsilon}} - \frac{1}{a}\Big)\mathrm{e}^{-\frac{tb}{a}}\Big| \\ &\leqslant C_\beta\frac{\varepsilon^\beta}{a(at)^\beta} + C\frac{\varepsilon}{a} + \frac{4\varepsilon b}{am_\varepsilon\sqrt{q_\varepsilon}}\mathrm{e}^{-\frac{tb}{a}},\end{aligned}$$

and this implies (18). □

Setting $\beta=0$, we obtain

Corollary 7. *There exists $C>0$ such that the estimate*

$$a(\lambda)|s_\varepsilon(t,\lambda)|\leqslant C\varepsilon \tag{19}$$

holds for all $\lambda\in\sigma$, $t\geqslant 0$ and $\varepsilon\in(0,\varepsilon_0)$.

Proof of Theorem 1. The estimate (4) is a direct consequence of (14), (15) and (19). $\square$

Proof of Proposition 2. The estimate (5) is a direct consequence of (4). To estimate $u_\varepsilon' - u_0'$, note that $u_\varepsilon' = v_\varepsilon$ is a solution of (1) with $f = 0$, $v(0) = u_{1\varepsilon}$, $v_t(0) = -\frac{1}{\varepsilon}(Bu_{0\varepsilon} + Au_{1\varepsilon})$ and that $u_0' = v_0$ is the solution of (2) with $f = 0$, $v(0) = -A^{-1}Bu_{00}$. Inserting these initial data into (5) and using $t\|BA^{-1}\mathrm{e}^{-tBA^{-1}}\| \leqslant 1$, we obtain (6). $\square$

Proof of Proposition 3. The estimate (7) is a direct consequence of (4), $\|\mathrm{e}^{-tBA^{-1}}\| \leqslant \mathrm{e}^{-t\gamma}$ and of

$$
\begin{aligned}
t\|BA^{-1}\mathrm{e}^{-tBA^{-1}}\| &\leqslant \sup_{\lambda\in\sigma} t\frac{b(\lambda)}{a(\lambda)}\mathrm{e}^{-\frac{tb(\lambda)}{a(\lambda)}} \\
&\leqslant \sup_{x\geqslant\gamma} tx\mathrm{e}^{-tx} \leqslant C_\delta \mathrm{e}^{-(\gamma-\delta)t}.
\end{aligned}
$$

$\square$

We end this Section by estimating the difference of the solutions of nonhomogeneous equations. Besides being of independent interest, this estimate is needed in the next Section.

Let $f_\varepsilon (\varepsilon \geqslant 0)$ be continuous X-valued functions and let u_ε $(\varepsilon > 0)$ be the mild solution (see [3]) of

$$\varepsilon u_{tt} + Au_t + Bu = f_\varepsilon, \quad u(0) = 0, \quad u_t(0) = 0 \tag{20}$$

and let u_0 be the solution of

$$Au_t + Bu = f_0, \ u(0) = 0. \tag{21}$$

Proposition 8. *Let $\beta \in (0,1)$. There exists $C > 0$ such that the estimate*

$$\|B^{1/2}(u_\varepsilon(t) - u_0(t))\| \leqslant C[\|f_\varepsilon - f_0\|_{L^1([0,t],X)} + (\varepsilon t + \varepsilon^\beta t^{1-\beta})\|f_0\|_{C([0,t],X)}] \tag{22}$$

holds for all $t \geqslant 0$ and $\varepsilon \in (0,\varepsilon_0)$.

Proof. The difference $u_\varepsilon - u_0$ is estimated using the identity

$$
\begin{aligned}
&u_\varepsilon(t) - u_0(t) \\
&\quad = \frac{1}{\varepsilon}\int_0^t S_\varepsilon(t-s)[f_\varepsilon(s) - f_0(s)]\,\mathrm{d}s + \int_0^t \Big[\frac{1}{\varepsilon}S_\varepsilon(t-s) - A^{-1}\mathrm{e}^{-(t-s)BA^{-1}}\Big] f_0(s)\,\mathrm{d}s
\end{aligned}
$$

where $S_\varepsilon(t) = s_\varepsilon(t, K)$. The first term is estimated using (19):

$$\begin{aligned}
&\int_0^t \Big\| \frac{1}{\varepsilon} B^{1/2} S_\varepsilon(t-s)[f_\varepsilon(s) - f_0(s)] \Big\| \,\mathrm{d}s \\
&\leqslant \|B^{1/2}A^{-1}\| \int_0^t \Big\| \frac{1}{\varepsilon} A S_\varepsilon(t-s) \Big\| \|f_\varepsilon(s) - f_0(s)\| \,\mathrm{d}s \leqslant C \int_0^t \|f_\varepsilon(s) - f_0(s)\| \,\mathrm{d}s,
\end{aligned}$$

and the second term by (18):

$$\begin{aligned}
&\int_0^t \Big\| B^{1/2} \Big[\frac{1}{\varepsilon} S_\varepsilon(t-s) - A^{-1} \mathrm{e}^{-(t-s)BA^{-1}} \Big] f_0(s) \Big\| \,\mathrm{d}s \\
&\leqslant C \|B^{1/2}A^{-1}\| \int_0^t (\varepsilon + \varepsilon^\beta \|A^{-\beta}\| (t-s)^{-\beta}) \|f_0(s)\| \,\mathrm{d}s.
\end{aligned}$$

This implies (22). □

If f_ε and f_0 are C^1-functions then we can apply (22) to the differentiated initial value problems to obtain an estimate for $\|B^{1/2}(u_\varepsilon'(t) - u_0'(t))\|$. The precise statement is omitted.

3. Proof of Theorem 4

The existence of a continuous $D(B^{1/2})$-valued global classical solution follows from Proposition 6 in [3] (with $Z = B^{1/2}$). We estimate

$$\|B^{1/2}(u_\varepsilon(t) - u_0(t))\| \leqslant \sum_{i=1}^{5} l_\varepsilon^{(i)}(t)$$

with

$$\begin{aligned}
l_\varepsilon^{(1)}(t) &= \|B^{1/2}[C_\varepsilon(t) - C_0(t)]u_{00}\|, \\
l_\varepsilon^{(2)}(t) &= \|B^{1/2}C_\varepsilon(t)(u_{0\varepsilon} - u_{00})\|, \\
l_\varepsilon^{(3)}(t) &= \|B^{1/2}S_\varepsilon(t)u_{1\varepsilon}\|, \\
l_\varepsilon^{(4)}(t) &= \int_0^t \Big\| B^{1/2} \Big[\frac{1}{\varepsilon} S_\varepsilon(t-s) - A^{-1}C_0(t-s) \Big] f(u_0(s)) \Big\| \,\mathrm{d}s, \\
l_\varepsilon^{(5)}(t) &= \frac{1}{\varepsilon} \int_0^t \|B^{1/2}S_\varepsilon(t-s)[f(u_\varepsilon(s)) - f(u_0(s))]\| \,\mathrm{d}s.
\end{aligned}$$

From (5) it follows that

$$l_\varepsilon^{(1)}(t) \leqslant C\varepsilon \|B^{1/2}u_{00}\|, \ l_\varepsilon^{(2)}(t) \leqslant C\|B^{1/2}(u_{0\varepsilon} - u_{00})\|$$

and

$$l_\varepsilon^{(3)}(t) \leqslant C\varepsilon\|u_{1\varepsilon}\|.$$

From (22) it follows that

$$l_\varepsilon^{(4)}(t) \leqslant C\|f(u_0)\|_{C([0,t],X)}(\varepsilon t + \varepsilon^\beta t^{1-\beta})$$

and

$$l_\varepsilon^{(5)}(t) \leqslant \int_0^t \|f(u_\varepsilon(s) - f(u_0(s))\| \,\mathrm{d}s \ .$$

Since $\|A^{1/2}u_\varepsilon(t)\|$ is bounded independently of ε and t by the energy inequality (see [3], p. 101), it follows from (9) that

$$l_\varepsilon^{(5)}(t) \leqslant C \int_0^t \|B^{1/2}(u_\varepsilon(s) - u_0(s))\| \,\mathrm{d}s.$$

Applying Gronwall's lemma we find (10).

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